MATH202X-F01/UX1
Spring 2015

Practice Midterm Exam 2

Name: Answer Key

1.

(a) Prove that $\displaystyle\lim_{(x,y)\to(1,1)} \frac{xy-1}{x+y^2-2}$ does not exist.

Solution: Consider the following paths:

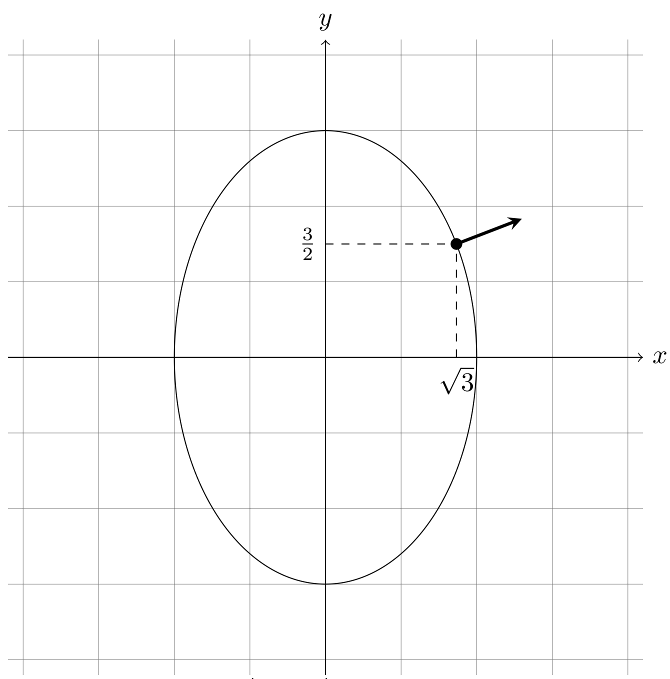
- $y=1$: $\displaystyle\lim_{(x,1)\to(1,1)} \frac{xy-1}{x+y^2-2} = \lim_{x\to 1}\frac{x-1}{x+1-2} = 1.$
- $x=1$: $\displaystyle\lim_{(1,y)\to(1,1)} \frac{xy-1}{x+y^2-2} = \lim_{y\to 1}\frac{y-1}{1+y^2-2} = \lim_{y\to 1}\frac{1}{y+1} = \frac{1}{2}.$

Since the two paths lead to different limits, the limit DNE.

(b) Consider the surface $z = \dfrac{x^2}{4} + \dfrac{y^2}{9}$ and the point $\left(\sqrt{3}, \frac{3}{2}, 1\right)$ on that surface. Using the axes and grid below,

- Draw the level curve to the surface going through that point.
- Sketch the gradient at that point. Briefly explain your reasoning.

Solution:



The gradient $\nabla z = \left\langle \dfrac{x}{2}, \dfrac{2y}{9} \right\rangle$ is always orthogonal to the level curve, and since the coordinates for the gradient are both positive at $\left(\sqrt{3}, \dfrac{3}{2}\right)$, it is oriented towards the outside of the level curve.

2. Suppose that $z = x^2 + 2xy - y^2$ where $x = 2u + v$ and $y = u - v$. Find z_u in two ways:

(a) using the multivariable chain rule.

Solution:

$$\begin{aligned} z_u &= z_x x_u + z_y y_u \\ &= (2x + 2y)(2) + (2x - 2y)(1) \\ &= 6x + 2y = 2(3x + y) \\ &= 2(3(2u + v) + u - v) \\ &= 2(6u + 3v + u - v) \\ &= \boxed{2(7u + 2v)} \end{aligned}$$

(b) using direct substitution.

Solution:

$$\begin{aligned} z_u &= ((2u + v)^2 + 2(2u + v)(u - v) - (u - v)^2)_u \\ &= 2(2)(2u + v) + 2(2(u - v) + (2u + v)(1)) - 2(u - v) \\ &= 8u + 4v + 2(2u - 2v + 2u + v) - 2u + 2v \\ &= 14u + 4v \\ &= \boxed{2(7u + 2v)} \end{aligned}$$

3. Find the equation of the tangent plane to the surface

$$3x \cos y - 2xz^2 + y^2 z = 1$$

at the point $(1, 0, 1)$.

Solution: Let $F(x, y, z) = 3x \cos y - 2xz^2 + y^2 z$. Then

$$\begin{aligned} \nabla F(x, y, z) &= \langle F_x, F_y, F_z \rangle \\ &= \left\langle 3 \cos y - 2z^2, -3x \sin y + 2yz, -4xz + y^2 \right\rangle, \\ \nabla F(1, 0, 1) &= \left\langle 3 \cos 0 - 2(1)^2, -3(1) \sin 0 + 2(0)(1), -4(1)(1) + (0)^2 \right\rangle \\ &= \langle 1, 0, -4 \rangle. \end{aligned}$$

So the equation of the tangent plane is:

$$1 \cdot (x - 1) + 0 \cdot (y - 0) - 4(z - 1) = 0 \quad \Longleftrightarrow \quad \boxed{x - 4z + 3 = 0}.$$

4. Find and classify the critical points of $z = 2xy - x^2y - \frac{1}{8}y^2$.

Solution: Critical points correspond to points where either $\nabla z = 0$ or is undefined. Here ∇z is defined everywhere, but

$$\begin{aligned} 0 = \nabla z \quad &\Longleftrightarrow \quad 0 = \left\langle 2y - 2xy, 2x - x^2 - \frac{1}{4}y \right\rangle \\ &\Longleftrightarrow \quad \begin{cases} 2y - 2xy = 0 \\ 2x - x^2 - \frac{y}{4} = 0 \end{cases} \\ &\Longleftrightarrow \quad \begin{cases} 2y(1-x) = 0 \quad ① \\ 2x - x^2 - \frac{y}{4} = 0 \quad ② \end{cases} \end{aligned}$$

From ①, we have:

- either $y = 0$, then from ②, either $x = 0$ or $x = 2$
- or $x = 1$, then from ②, $y = 4$.

So critical points are $(0, 0, 0)$, $(2, 0, 0)$, $(1, 4, 2)$. To classify them, we use the second partials test, and thus compute at each of the critical points the Jacobian:

$$d = z_{xx}z_{yy} - z_{xy}^2.$$

Since we have:

$$\begin{aligned} z_{xx} &= -2y, \\ z_{yy} &= -\frac{1}{4}, \\ z_{xy} &= 2(1-x), \end{aligned}$$

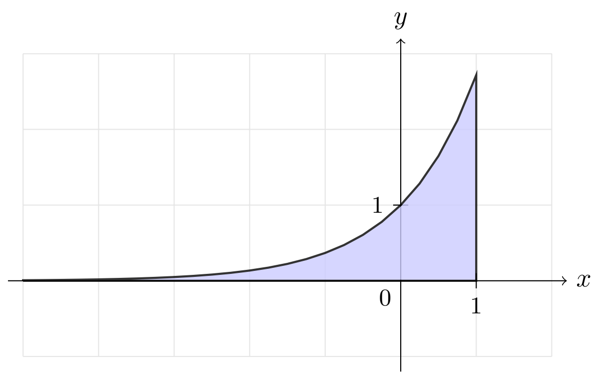
then:

$$\begin{aligned} d &= \frac{y}{2} - 4(1-x)^2, \\ d|_{(0,0)} &= -4 < 0 \quad &\Longrightarrow \quad &\boxed{(0,0,0) \text{ is a saddle point}}, \\ d|_{(2,0)} &= -4 < 0 \quad &\Longrightarrow \quad &\boxed{(2,0,0) \text{ is a saddle point}}, \\ d|_{(1,4)} &= 2 > 0 \text{ and } z_{xx} < 0 \quad &\Longrightarrow \quad &\boxed{(1,4,2) \text{ is a relative maximum}}. \end{aligned}$$

5. Compute the iterated integral:

$$I = \int_0^e \int_{\ln y}^1 \sin\left(ye^{-x}\right)\, dx\, dy.$$

Solution: As it is, we cannot integrate directly, so we need to change the order of integration. The region of integration is as follows:



So the integral becomes:

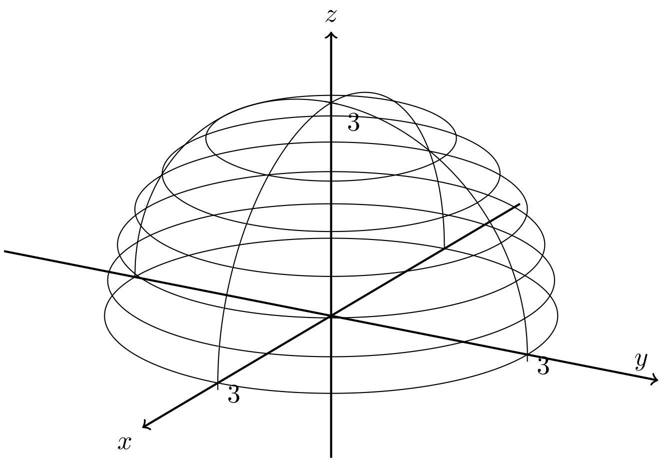
$$\begin{aligned} I &= \int_{-\infty}^1 \int_0^{e^x} \sin\left(ye^{-x}\right)\, dy\, dx \\ &= \int_{-\infty}^1 \left[\frac{-1}{e^{-x}} \cos\left(ye^{-x}\right)\right]_0^{e^x} dx \\ &= \int_{-\infty}^1 \left[-e^x \cos\left(e^x e^{-x}\right) + e^x \cos 0\right] dx \\ &= \int_{-\infty}^1 \left[-e^x \cos 1 + e^x\right] dx \\ &= \int_{-\infty}^1 e^x (1 - \cos 1)\, dx \\ &= \left[e^x (1 - \cos 1)\right]_{-\infty}^1 \\ &= e^1 (1 - \cos 1) - 0 \\ &= \boxed{e(1 - \cos 1).} \end{aligned}$$

6. The total mass of a solid is given by:

$$m = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{0}^{\sqrt{9-x^2-y^2}} z\sqrt{x^2+y^2+z^2}\, dz\, dy\, dx.$$

(a) Describe and sketch the solid in space.

Solution:



The solid is the $\boxed{\text{top half of a sphere of radius 3 centered at the origin}}$.

(b) Switch the integral to spherical coordinates. You need not evaluate it but you may choose to do so for extra credit.

Solution:

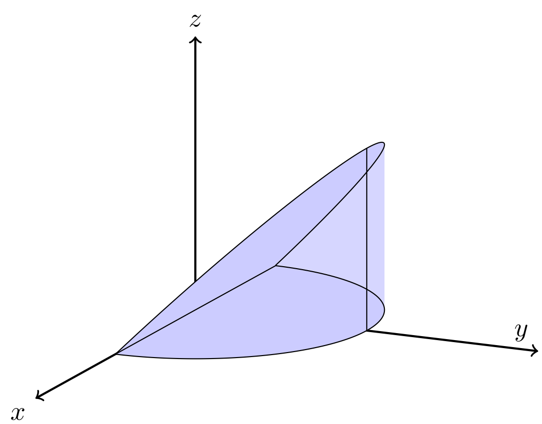
$$m = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^3 (\rho\cos\phi)(\rho)\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$\boxed{m = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^3 \rho^4 \cos\phi \sin\phi \, d\rho\, d\phi\, d\theta.}$$

<u>Extra credit</u>:

$$\begin{aligned}
m &= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^3 \rho^4 \cos\phi \sin\phi \, d\rho\, d\phi\, d\theta \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \left[\frac{\rho^5}{5}\cos\phi\sin\phi\right]_0^3 d\phi\, d\theta \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \frac{3^5}{10}\sin(2\phi)\, d\phi\, d\theta \\
&= \int_0^{2\pi} \left[-\frac{3^5}{20}\cos(2\phi)\right]_0^{\frac{\pi}{2}} d\theta \\
&= \int_0^{2\pi} \left(-\frac{81(3)}{20}\cos\pi + \frac{81(3)}{20}\cos 0\right) d\theta \\
&= \frac{243}{10}\int_0^{2\pi} d\theta \\
&= \boxed{\frac{243\pi}{5}.}
\end{aligned}$$

7. Set up a triple integral to calculate the volume of the solid (illustrated below) bounded by the cylinder $x^2 + y^2 = 1$, the plane $y = z$ and the xy-plane. You may use the coordinate system of your choice. Then evaluate the integral.



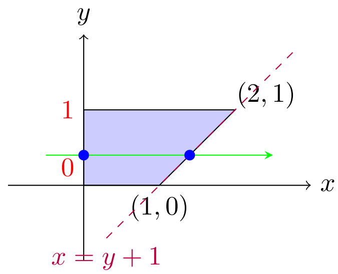
Solution: Cylindrical coordinates are the easiest here:

$$\boxed{V = \int_0^{\pi} \int_0^1 \int_0^{r \sin\theta} r\, dz\, dr\, d\theta.}$$

$$\begin{aligned}
V &= \int_0^{\pi} \int_0^1 \int_0^{r \sin\theta} r\, dz\, dr\, d\theta \\
&= \int_0^{\pi} \int_0^1 [zr]_0^{r\sin\theta}\, dr\, d\theta \\
&= \int_0^{\pi} \int_0^1 r^2 \sin\theta\, dr\, d\theta \\
&= \int_0^{\pi} \left[\frac{r^3}{3}\sin\theta\right]_0^1 d\theta \\
&= \int_0^{\pi} \frac{1}{3}\sin\theta\, d\theta \\
&= \left[-\frac{1}{3}\cos\theta\right]_0^{\pi} \\
&= -\frac{1}{3}\cos\pi + \frac{1}{3}\cos 0 \\
&= \boxed{\frac{2}{3}.}
\end{aligned}$$

8. Consider the following lamina with density $\rho = 2x$.

Solution:



(a) Find the total mass m of the lamina.

Solution:

$$\begin{aligned} m &= \int_0^1 \int_0^{y+1} 2x \, dx \, dy \\ &= \int_0^1 \left[x^2\right]_0^{y+1} \, dy \\ &= \int_0^1 (y+1)^2 \, dy \\ &= \left[\frac{1}{3}(y+1)^3\right]_0^1 \\ &= \frac{1}{3}(8-1) \\ &= \boxed{\frac{7}{3}.} \end{aligned}$$

(b) If the first moment about the x-axis is $\frac{17}{12}$, and the first moment about the y-axis is $\frac{5}{2}$, where is the center of mass $(\bar{x}, \bar{y})$ of the lamina?

Solution:

$$\begin{aligned} (\bar{x}, \bar{y}) &= \left(\frac{M_y}{m}, \frac{M_x}{m}\right) \\ &= \left(\frac{\frac{5}{2}}{\frac{7}{3}}, \frac{\frac{17}{12}}{\frac{7}{3}}\right) \\ &= \boxed{\left(\frac{15}{14}, \frac{17}{28}\right).} \end{aligned}$$