Math F253 Midterm 2 Fall 2021

Name: Solutions

Section: F02 (Maxwell)

Student Id: ____________________

Calculator Model: ____________________

Rules:

You have 60 minutes to complete the exam.

Partial credit will be awarded, but you must show your work.

A scientific or graphing calculator (without symbolic manipulation) is allowed.

A one page sheet of paper (8 1/2 in. x 11 in.) with handwritten notes on one side is allowed.

No other aids are permitted.

Place a box around your $\boxed{\text{FINAL ANSWER}}$ to each question where appropriate.

If you need extra space, you can use the back sides of the pages. Please make it obvious when you have done so.

Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Extra Credit	3	
Total	60	

1. (10 points)

The temperature in the x-y plane describing the floor of a room is given by

$$T(x, y) = 10(1 + e^{-x^2-y^2})$$

where x and y are measured in meters and T is measured in degrees Celcius.

a. (3 points) Compute $\vec{\nabla}T$.

$$\vec{\nabla}T = \left\langle \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y} \right\rangle$$

$$= \left\langle -20xe^{-x^2-y^2}, -20ye^{-x^2-y^2} \right\rangle$$

b. (4 points) Suppose you are walking along the floor. At time $t = 0$ seconds you are at $P(1/2, 1)$ and that your position is changing at $t = 0$ with $dx/dt = 1$ m/s and $dy/dt = 3$ m/s. What is the rate of change of temperature at $t = 0$? Be sure to include units in your answer.

$\vec{\nabla}T$ at P is $-20e^{-\frac{1}{4}-1}\left\langle \frac{1}{2}, 1 \right\rangle = e^{-5/4}\langle -10, -20\rangle$

$$\frac{dT}{dt} = \vec{\nabla}T \cdot \vec{v} = \vec{\nabla}T \cdot \langle 1, 3\rangle = e^{-5/4}(-10-60)$$

$$= -70e^{-5/4} \text{ °C/s}$$

c. (3 points) At the point $P(1/2, 1)$, the same point as in part **(b)**, in what direction is the temperature increasing the fastest? Express your answer as a unit vector.

$\vec{\nabla}T$ is parallel to $\langle -1, -2\rangle$

unit vector: $\vec{u} = \frac{1}{\sqrt{1^2+(-2)^2}}\langle -1, -2\rangle = \left\langle -\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right\rangle$

2. (10 points)

A simple electrical circuit contains a battery and a variable resistor. The voltage V of the battery, the current I in amperes (amps) and the resistance R in ohms (Ω) are related by Ohm's Law

$$V = IR.$$

Initially, $R = 6\ \Omega$ and $V = 12$ volts and therefore the current is 2 amps. Use either the linearization or differentials (your choice, you'll get the same result!) to estimate the change in the current if the battery charge goes down by 0.1 volts and the resistance goes up by 0.2 Ω. Note that an amp is the same thing as a volt per ohm. Express your answer with units. Hint: You may wish to solve for I first.

$$I = \frac{V}{R}$$

$$dI = \frac{dV}{R} - \frac{V}{R^2}\,dR$$

$$= \frac{-0.1}{6} - \frac{12}{36}\,0.2$$

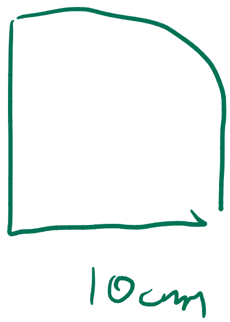
$$= -\frac{1}{6}\cdot\frac{1}{10} - \frac{1}{3}\cdot\frac{2}{10}$$

$$= -\frac{5}{60} = \boxed{-\frac{1}{12}\ \text{amps}}$$

3. (10 points)

A thin plate of metal has the shape of a quarter circle in the first quadrant of the plane with center at the origin and radius 10cm. It has a mass density given by $\rho(x, y) = y/5$ g/cm^2.

a. (4 points) Compute the mass of the plate. Units, please!



$$\int_0^{\pi/2}\int_0^{10} \frac{r\sin\theta}{5}\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \frac{r^3}{3}\Big|_0^{10} \frac{\sin\theta\,d\theta}{5}$$

$$= \frac{10^3}{3}\,\frac{-\cos(\theta)}{5}\Big|_0^{\pi/2} = \frac{2}{3}\cdot 100$$

b. (4 points) Set up the integral for the moment of mass about the y axis, M_y, of the plate. For full credit, you should express your answer as an iterated integral in **polar** coordinates. DO NOT COMPUTE THE INTEGRAL!

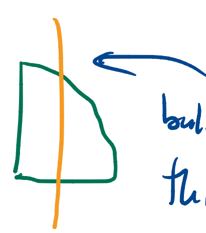
$$\int_0^{\pi/2}\int_0^{10} \frac{r^3\sin\theta\cos\theta}{5}\,dr\,d\theta$$

250

c. (2 points) In fact, M_y = 1250g cm. You now know have enough information to compute something about the center of mass of the plate. Compute this value and state what it represents.

$$\bar{x} = \frac{250}{\frac{2}{3}\cdot 100} = \frac{750}{200} = 3.75 \leftarrow$$

x-coord of center of mass



balances on this line

3. (10 points)

A thin plate of metal has the shape of a quarter circle in the first quadrant of the plane with center at the origin and radius 10cm. It has a mass density given by $\rho(x, y) = y/5$ g/cm^2.

a. (4 points) Compute the mass of the plate. Units, please!

$$\int_0^{\pi/2} \int_0^{10}$$

b. (4 points) Set up the integral for the moment of mass about the y axis, M_y, of the plate. For full credit, you should express your answer as an iterated integral in **polar** coordinates. DO NOT COMPUTE THE INTEGRAL!

c. (2 points) In fact, $M_y = 1250$g cm. You now know have enough information to compute something about the center of mass of the plate. Compute this value and state what it represents.

4. (10 points)

a. (6 points) Find the critical points of $f(x, y) = 2x^2y + 4x + y^2$.

$$\vec{\nabla} f = \langle 4xy + 4,\ 2x^2 + 2y \rangle$$

$$\vec{\nabla} f = 0 \Rightarrow 4xy = -4 \Rightarrow xy = -1$$

$$2x^2 + 2y = 0 \Rightarrow y = -x^2$$

$$y = -x^2 \Rightarrow x(-x^2) = -1 \Rightarrow x^3 = 1 \Rightarrow x = 1$$

$$\Rightarrow y = -1$$

One crit pt: $(1, -1)$

b. (4 points) You should have found that one of the critical points has a ~~negative~~ positive x-coordinate. Classify, with justification, this critical point as a local min, local max or saddle point

$$\frac{\partial^2 f}{\partial x^2} = 4y \quad \frac{\partial^2 f}{\partial x \partial y} = 4x \quad \frac{\partial^2 f}{\partial y^2} = 2$$

$$\boxed{\text{Saddle}}$$

$$\left| \begin{bmatrix} 4y & 4x \\ 4x & 2 \end{bmatrix} \right| = 8y - 16x^2$$

$$= 8 \cdot (-1) - 16 = -24 < 0$$

5. (10 points)

a. (7 points) Use the method of Lagrange multipliers to max ~~mini~~mize $f(x, y) = xy$ subject to the constrant $g(x, y) = x + 2y = 5$.

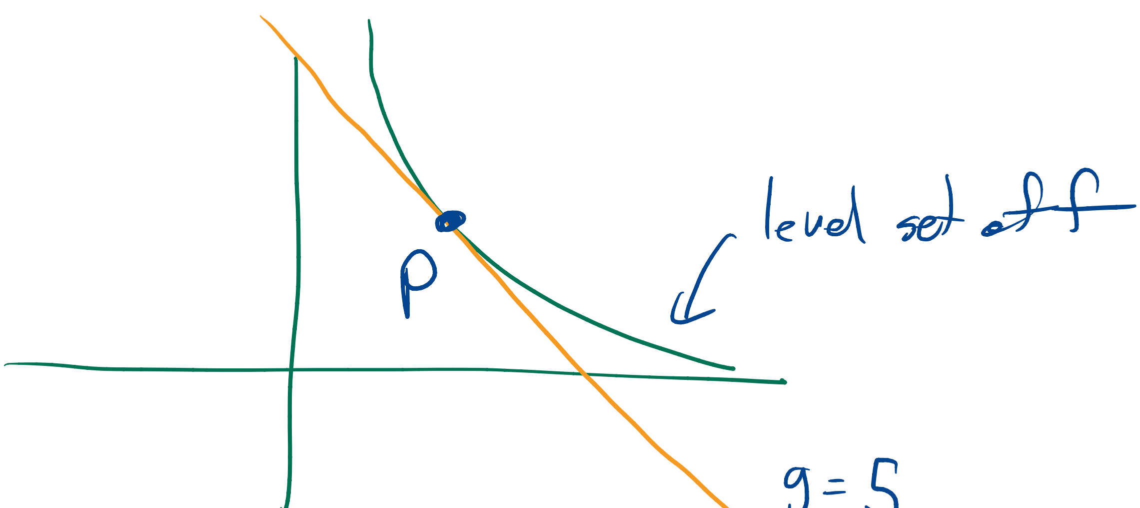
$$\vec{\nabla} f = \lambda \vec{\nabla} g \Rightarrow \langle y, x \rangle = \lambda \langle 1, 2 \rangle$$

$$g = 5$$

$$\left.\begin{aligned} y &= \lambda \\ x &= 2\lambda \\ x + 2y &= 5 \end{aligned}\right\} \Rightarrow 2\lambda + 2\lambda = 5 \qquad \lambda = 5/4$$

$$\boxed{x = 5/2 \qquad y = 5/4}$$

b. (3 points) The equations for the method of Lagrange multipliers imply that the gradient of f points along the same line as the gradient of g at the minimum point. What does this imply about the level sets of f and g at the minimum point? Make a **rough** sketch below of the two level sets that contains the point P where the minimum value occurs to illustrates this idea. Your sketch should label the level set of f, the level set of g and the point P.



6. (10 points)

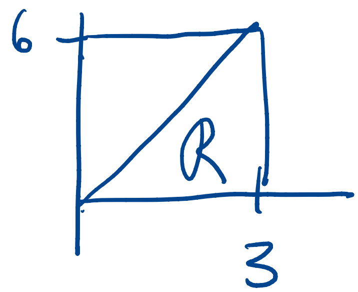
The iterated integral

$$\int_0^6 \int_{y/2}^3 e^{x^2}\, dx\, dy$$

represents a double integral over a region $\mathcal{R}$ in the plane.

a. (3 points) Make a careful sketch of the region $\mathcal{R}$ below.

$x = \frac{y}{2} \Rightarrow y = 2x$



b. (7 points) Compute the integral by interchanging the order of integration.

$$\int_0^3 \int_0^{2x} e^{x^2}\, dy\, dx = \int_0^3 2x e^{x^2}\, dx$$

$u = x^2 \quad du = 2x\, dx$

$$= \int_0^9 e^u\, du$$

$$= \boxed{e^9}$$

7. (Extra Credit: 3 points)

Compute the integral in problem **3b**.

$$\int_0^{\pi/2}\int_0^{10} \frac{r^3 \sin\theta\cos\theta}{5}\, dr\, d\theta$$

$$\int_0^{\pi/2} \frac{r^4}{20}\Big|_0^{10} \sin\theta\cos\theta\, d\theta$$

$$\frac{10^4}{20}\int_0^{\pi/2} \sin\theta\cos\theta\, d\theta = \frac{10^3}{2}\cdot\frac{1}{2} = 250$$