MATH253X-UX1
Spring 2016

Midterm Exam 1

Name: Answer Key

Instructions. You have 60 minutes. No notes, book, or calculators allowed. *Show all your work* in order to receive full credit.

1. Consider a triangle in space with vertices $A(1,0,-1)$, $B(2,2,-2)$, and $C(-2,1,0)$.

(a) Find the equation of the sphere centered at A and going through B.

Solution: Radius of the sphere: $\left\|\overrightarrow{AB}\right\| = \sqrt{(2-1)^2+(2-0)^2+(-2+1)^2} = \sqrt{1+4+1} = \sqrt{6}$. So the equation of the sphere is:

$$\boxed{(x-1)^2+y^2+(z+1)^2=6}.$$

(b) Find the cosine of the angle at vertex A in the triangle.

Solution:

$$\begin{aligned}
\cos\alpha &= \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\left\|\overrightarrow{AB}\right\|\left\|\overrightarrow{AC}\right\|} = \frac{\langle 2-1, 2-0, -2+1\rangle\cdot\langle -2-1, 1-0, 0+1\rangle}{\sqrt{6}\sqrt{(-2-1)^2+(1-0)^2+(0+1)^2}} \\
&= \frac{\langle 1,2,-1\rangle\cdot\langle -3,1,1\rangle}{\sqrt{6}\sqrt{9+1+1}} = \frac{1(-3)+2(1)-1(1)}{\sqrt{6}\sqrt{11}} \\
&= \frac{-2}{\sqrt{66}} = \boxed{-\frac{\sqrt{66}}{33}}
\end{aligned}$$

(c) Find the equation of the plane containing the triangle ABC.

Solution: A normal vector to the plane is (any scalar multiple of):

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -3 & 1 & 1 \end{vmatrix} = \langle 2(1)-1(-1), -(1(1)+3(-1)), 1(1)+3(2)\rangle = \langle 3,2,7\rangle\,.$$

And so the equation of the plane is:

$$\boxed{3(x-1)+2y+7(z+1)=0} \quad \text{or equivalently} \quad \boxed{3x+2y+7z+4=0}.$$

2. Let $\mathbf{r}(t) = \langle 3\cos t, 5\sin t, -4\cos t\rangle$ describe the trajectory of a particle over time, where position is measured in meters and time in seconds.

(a) Find the distance traveled (i.e. the arc length) from $t=0$ s to $t=2\pi$ s.

Solution: We have:

$$\begin{aligned}
&\mathbf{r}'(t) = \langle -3\sin t, 5\cos t, 4\sin t\rangle \\
\Rightarrow\quad &\|\mathbf{r}'(t)\| = \sqrt{9\sin^2 t+25\cos^2 t+16\sin^2 t} = \sqrt{25\sin^2 t+25\cos^2 t} = \sqrt{25} = 5 \text{ m/s.}
\end{aligned}$$

So the distance traveled is:

$$s = \int_0^{2\pi}\|\mathbf{r}'(t)\|\,dt = \int_0^{2\pi} 5\,dt = \Big[5t\Big]_0^{2\pi} = \boxed{10\pi \text{ m}}.$$

(b) Show that the trajectory $\mathbf{r}(t) = \langle 3\cos t, 5\sin t, -4\cos t\rangle$ sits on both surfaces $4x + 3z = 0$ and $\frac{x^2}{9} + \frac{y^2}{25} = 1$ at all times.

Solution: Since $x = 3\cos t$, $y = 5\sin t$, and $z = -4\cos t$, then for all t:

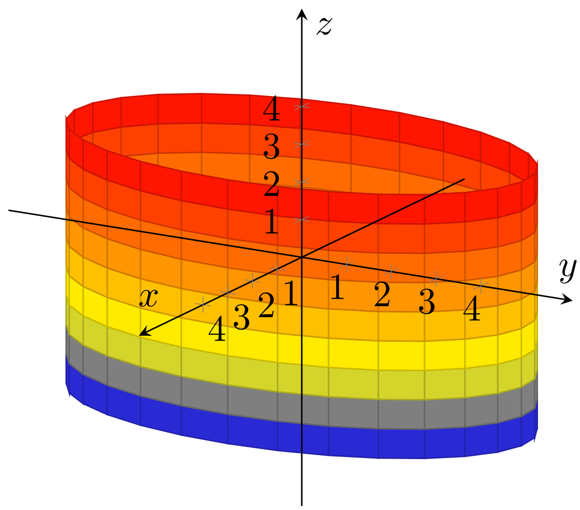
$$4x + 3z = 4(3\cos t) + 3(-4\cos t) = 12\cos t - 12\cos t = 0 \quad \checkmark$$

and

$$\frac{x^2}{9} + \frac{y^2}{25} = \frac{(3\cos t)^2}{9} + \frac{(5\sin t)^2}{25} = \frac{9\cos^2 t}{9} + \frac{25\sin^2 t}{25} = \cos^2 t + \sin^2 t = 1 \quad \checkmark$$

(c) Sketch the surface $\frac{x^2}{9} + \frac{y^2}{25} = 1$. Include a scale.

Solution:

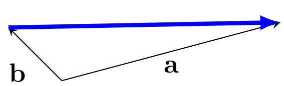


3. Consider vectors **a** and **b** as shown below.

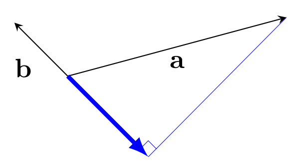
(a) Sketch the following:

Solution:

$\mathbf{a} - \mathbf{b}$



$\text{proj}_{\mathbf{b}}\, \mathbf{a}$



(b) State whether the following statements are true or false. Briefly justify.

- $\mathbf{a} \cdot \mathbf{b} \geq 0$ **False. The angle between them is obtuse so cosine is negative.**
- $\mathbf{a} \times \mathbf{b}$ points out of the page towards you. **True, by the right hand rule.**

4. Let a space curve be described by $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 3t\mathbf{k}$.

(a) Find the symmetric equations of the tangent line to the curve at the point $P(-1, 1, -3)$.

Solution:

- $\overrightarrow{OP} = \mathbf{r}(-1)$
- $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 1, 2t, 3\rangle$
- $\mathbf{v}(-1) = \mathbf{r}'(-1) = \langle 1, -2, 3\rangle$
- so symmetric equations for the tangent line are:

$$\boxed{x + 1 = \frac{y-1}{-2} = \frac{z+3}{3}}$$

(b) Find the tangential component of the acceleration for any t.

Solution: We have

$$\|\mathbf{v}(t)\| = \sqrt{1 + 4t^2 + 9} = \sqrt{4t^2 + 10}$$

so the tangential component of acceleration is:

$$a_{\mathbf{T}} = \|\mathbf{v}\|' = \boxed{\frac{4t}{\sqrt{4t^2 + 10}}}.$$

Or using another form:

$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \frac{\mathbf{a} \cdot \mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle 0, 2, 0\rangle \cdot \langle 1, 2t, 3\rangle}{\sqrt{4t^2 + 10}} = \boxed{\frac{4t}{\sqrt{4t^2 + 10}}}$$

5. Consider the line given by:

$$x = 2 + t \quad , \quad y = 1 - t \quad , \quad z = 5 - 4t.$$

(a) Show that the line is parallel to but not in the plane $x - 3y + z = 1$.

Solution: The line is parallel to the plane if its direction is orthogonal to the normal vector. Here:

$$\langle 1, -1, -4\rangle \cdot \langle 1, -3, 1\rangle = 1(1) - 1(-3) - 4(1) = 1 + 3 - 4 = 0 \quad \checkmark$$

To show it's not in the plane, we need only one point from the line (since parallel already) to plug into the equation of the plane:

$$2 - 3(1) + 5 = 4 \neq 1.$$

Or from the start, you can plug in the coordinates for the whole line into the plane:

$$(2 + t) - 3(1 - t) + (5 - 4t) = 2 + t - 3 + 3t + 5 - 4t = 4 \neq 1$$

for any t so there is no intersection between the line and the plane, which will only happen if the line is parallel to the plane but not in it.

(b) Find the distance from the line to the plane. *Hint:* You can use any point from the line.

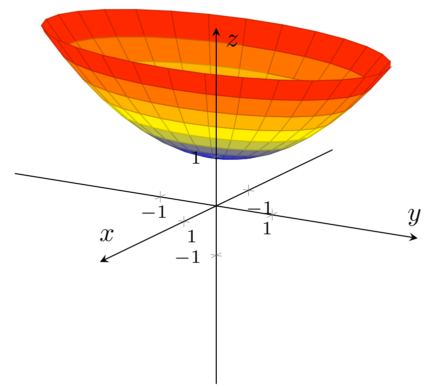
Solution: Rewrite the plane as $x - 3y + z - 1 = 0$, then using the point $(2, 1, 5)$ from the line, the distance from the line to the plane is:

$$d = \frac{|2 - 3(1) + 5 - 1|}{\|\langle 1, -3, 1\rangle\|} = \frac{3}{\sqrt{1 + 9 + 1}} = \boxed{\frac{3\sqrt{11}}{11}}.$$

6. Identify the surface from its equation then sketch the surface.

(a) $z = 1 + x^2 + \dfrac{y^2}{4}$

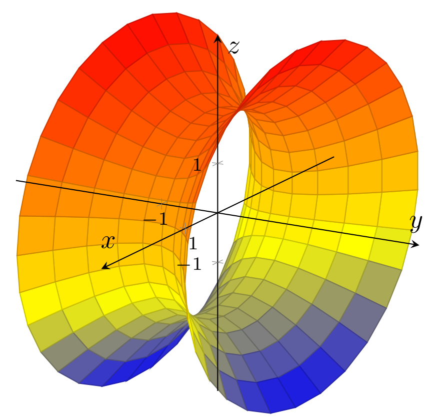
Solution:



Type of surface: elliptic paraboloid

(b) $x^2 + z^2 - 4y^2 = 4$

Solution:



Type of surface: hyperboloid of one sheet

7. A particle is moving in space from an initial position $\mathbf{r}(0) = \langle 0, 0, 1\rangle$ and initial velocity $\mathbf{v}(0) = \langle 2, 1, 2\rangle$ according to the following ***acceleration*** (measured in ft/s) at time t:

$$\mathbf{a}(t) = \left\langle 4e^{2t}, 6t, \frac{1}{(t+1)^2} \right\rangle \qquad , \qquad t \geq 0.$$

Find the position of the particle at $t = 1$ s.

Solution:

- $\mathbf{v}(t) = \displaystyle\int \mathbf{a}(t)\, dt = \left\langle 2e^{2t}, 3t^2, \frac{-1}{t+1} \right\rangle + \mathbf{c}$
- $\langle 2, 1, 2\rangle = \mathbf{v}(0) = \langle 2, 0, -1\rangle + \mathbf{c} \quad \Rightarrow \quad \mathbf{c} = \langle 2-2, 1-0, 2+1\rangle = \langle 0, 1, 3\rangle$
- $\mathbf{v}(t) = \left\langle 2e^{2t}, 3t^2 + 1, \dfrac{-1}{t+1} + 3 \right\rangle$
- for the position at $t = 1$ s, we can use:

$$\mathbf{r}(1) - \mathbf{r}(0) = \int_0^1 \mathbf{v}(t)\, dt = \int_0^1 \left\langle 2e^{2t}, 3t^2 + 1, \frac{-1}{t+1} + 3 \right\rangle dt = \Big[\left\langle e^{2t}, t^3 + t, 3t - \ln(t+1) \right\rangle \Big]_0^1$$

$$\iff \quad \mathbf{r}(1) - \langle 0, 0, 1\rangle = \left\langle e^2, 2, 3 - \ln(2) \right\rangle - \langle 1, 0, 0\rangle$$

$$\iff \quad \mathbf{r}(1) = \left\langle e^2 + 0 - 1, 2 + 0 - 0, 3 - \ln(2) + 1 - 0 \right\rangle \quad \iff \quad \boxed{\mathbf{r}(1) = \left\langle e^2 - 1, 2, 4 - \ln(2) \right\rangle.}$$

8. Let $f(t) = \dfrac{1}{t}$ and let $\mathbf{r}(t) = \left\langle t^2 - 1, \tan t \right\rangle$. Compute the derivative $\dfrac{d}{dt}\left[f(t)\mathbf{r}(t)\right]$ by using the rules of differentiation for a product. No credit will be given for substituting first.

Solution: We have $f'(t) = \dfrac{-1}{t^2}$ and $\mathbf{r}'(t) = \langle 2t, \sec^2 t \rangle$.
So, by the product rule

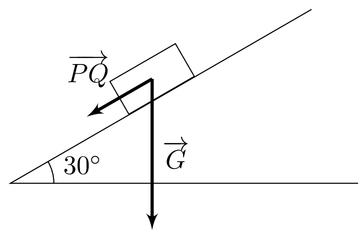
$$\frac{d}{dt}\left[f(t)\mathbf{r}(t)\right] = f'(t)\mathbf{r}(t) + f(t)\mathbf{r}'(t) = \frac{-1}{t^2}\left\langle t^2 - 1, \tan t\right\rangle + \frac{1}{t}\left\langle 2t, \sec^2 t\right\rangle = \left\langle -1 + \frac{1}{t^2}, \frac{-\tan t}{t^2}\right\rangle + \left\langle 2, \frac{\sec^2 t}{t}\right\rangle$$

$$\boxed{\frac{d}{dt}\left[f(t)\mathbf{r}(t)\right] = \frac{1}{t^2}\left\langle t^2 + 1, t\sec^2 t - \tan t\right\rangle}$$

9. Physical applications. Choose *one* of the following problems to solve. You may do the other for extra credit only.

(a) Find the work done by gravity if a child on a sled with combined weight 60 lbs goes down 50 feet along a 30° incline.

Solution:



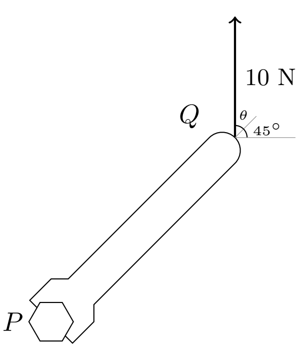
The angle between $\overrightarrow{PQ}$ and the weight from gravity $\overrightarrow{G}$ is 60°. And so we have:

$$W = \overrightarrow{G} \cdot \overrightarrow{PQ} = \left\|\overrightarrow{G}\right\| \left\|\overrightarrow{PQ}\right\| \cos 60^\circ = 60(50)\left(\frac{1}{2}\right) = \boxed{1500 \text{ ft-lbs}}.$$

Alternately, set up $\overrightarrow{G} = < 0, -60 >$ and $\overrightarrow{PQ} = \langle 50\cos 210^\circ, 50\sin 210^\circ\rangle = \left\langle -25\sqrt{3}, -25\right\rangle$ and do the dot product from the components.

(b) Find the magnitude of the torque when applying a force of 10 N directly upwards on a 20-cm wrench that makes a 45° angle with the horizontal.

Solution:



The magnitude of the torque is:

$$\tau = \left\|\overrightarrow{PQ} \times \overrightarrow{F}\right\| = \left\|\overrightarrow{PQ}\right\| \left\|\overrightarrow{F}\right\| \sin\theta = 0.2(10)\sin 45^\circ = 2\frac{\sqrt{2}}{2} = \boxed{\sqrt{2} \text{ Nm}}.$$

Alternately, set up $\overrightarrow{F} = \langle 0, 10, 0\rangle$ and $\overrightarrow{PQ} = \langle 0.2\cos 45°, 0.2\sin 45°, 0\rangle = \left\langle \frac{\sqrt{2}}{10}, \frac{\sqrt{2}}{10}, 0 \right\rangle = \frac{\sqrt{2}}{10}\langle 1, 1, 0\rangle$

then,

$$\overrightarrow{\tau} = \overrightarrow{PQ} \times \overrightarrow{F} = \frac{\sqrt{2}}{10}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 10 & 0 \end{vmatrix} = \frac{\sqrt{2}}{10}\langle 0, 0, 10\rangle = \sqrt{2} < 0, 0, 1 >= \sqrt{2}\mathbf{k}$$

and so the norm of the torque is $\sqrt{2}$ Nm.