# Math F253 Final Exam Fall 2021

**Name:** Solutions **Section: F02 (Maxwell)**

**Student Id:** ____________________

**Calculator Model:** ____________________

## Rules:

You have 60 minutes to complete the exam.

Partial credit will be awarded, but you must show your work.

A scientific or graphing calculator (without symbolic manipulation) is allowed.

A one page sheet of paper (8 1/2 in. x 11 in.) with handwritten notes on one side is allowed.

No other aids are permitted.

Place a box around your FINAL ANSWER to each question where appropriate.

If you need extra space, you can use the back sides of the pages. Please make it obvious when you have done so.

Turn off anything that might go beep during the exam.

Good luck!

| Problem | Possible | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 10 | |
| 7 | 10 | |
| 8 | 10 | |
| 9 | 10 | |
| 10 | 10 | |
| Extra Credit | 4 | |
| Total | 100 | |

**1. (10 points)**

Consider a pressure distribution $p(x, y) = xe^{-x^2-y}$ measured in Pascals (i.e., N/m$^2$) where position is measured in m.

**a. 5 points** Compute the gradient $\nabla p$. Include units in your answer.

$$\frac{\partial p}{\partial x} = e^{-x^2-y} - 2x^2 e^{-x^2-y} = e^{-x^2-y}(1-2x^2)$$

$$\frac{\partial p}{\partial y} = -x e^{-x^2-y}$$

$$\boxed{\vec{\nabla} p = e^{-x^2-y}\langle 1-2x^2, -x\rangle \frac{N}{m^3}}$$

**b. 5 points** Suppose at one moment of time you start at position $(x, y) = (2, 1)$ and travel with velocity $\mathbf{v} = \langle 1, 1\rangle$ m/s. What rate of change of pressure do you observe? Include units in your answer.

at $(2,1)$ $\vec{\nabla} p = e^{-2^2-1}\langle 1-2\,2^2, -2\rangle$

$$= e^{-5}\langle -7, -2\rangle$$

rate of change:

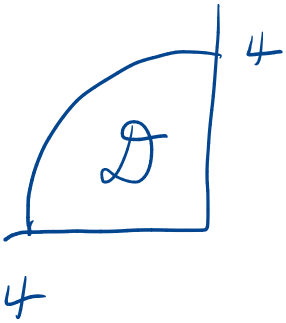
$$\vec{\nabla} p \cdot \vec{v} = e^{-5}\langle -7,-2\rangle\langle 1,1\rangle = -9e^{-5}\frac{N}{m^2 s}$$

$$= \boxed{-9e^{-5}\ Pa/s}$$

**2. (10 points)**

A thin plate is a quarter circle of radius 4cm has its vertex at the origin and lies in the region of the plane with $x < 0$ and $y > 0$. Its mass density is given by $2\sqrt{x^2 + y^2}$ g/cm$^2$ in this coordinate system.

**a. 7 points** Compute the mass of the plate.




$$\iint_D \rho \, dA = \int_{\pi/2}^{\pi} \int_0^4 2r \; r \, dr \, d\theta$$

$$= \int_{\pi/2}^{\pi} \int_0^4 2r^2 \, dr \, d\theta$$

$$= \int_{\pi/2}^{\pi} \frac{2}{3} r^3 \Big|_0^4 \, d\theta = \frac{\pi}{2} \cdot \frac{2}{3} 4^3$$

$$= \boxed{\frac{64\pi}{3}}$$

**b. 3 points** Write down an iterated integral that could be used to compute the $x$-component of the center of mass.

$$x = r\cos\theta$$

$$\bar{x} = \frac{M_y}{\text{Mass}} = \frac{3}{64\pi} \iint_D x \rho \, dA = \frac{3}{64\pi} \int_{\frac{\pi}{2}}^{\pi} \int_0^4 2r\cos\theta \, r^2 \, dr \, d\theta$$

$$= \boxed{\frac{3}{64\pi} \int_{\pi/2}^{\pi} \int_0^4 2r^3 \cos\theta \, dr \, d\theta}$$

**3. (10 points)**

A particle has acceleration $\mathbf{a}(t) = \langle \sin(t), \cos(t), -1 \rangle$ m/s$^2$.

**a. 5 points** Assuming the initial velocity is $\mathbf{v}(0) = \langle -3, 1, 2 \rangle$, compute the the particle's velocity $\mathbf{v}(t)$.

$$\vec{v} = \int \vec{a}\, dt + \vec{c}$$

$$= \langle -\cos(t), \sin(t), -t \rangle + \vec{c}$$

$$\vec{v}(0) = \langle -1, 0, 0 \rangle + \vec{c}$$

$$\Rightarrow \vec{c} = \langle -3, 1, 2 \rangle - \langle -1, 0, 0 \rangle = \langle -2, 1, 2 \rangle$$

$$\boxed{\vec{v}(t) = \langle -2 - \cos(t), 1 + \sin(t), 2 - t \rangle}$$

**b. 5 points** The same particle experiences a constant force $\mathbf{F} = \langle 0, 1, 2 \rangle$ N as it travels along this path. Compute the work done by the force from $t = 0$ to $t = \pi$ seconds.

[Hint: Although you may be tempted to try to compute the position $\mathbf{r}(t)$, this is not necessary!]

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi} \underbrace{\vec{F}(\vec{r}(t))}_{\langle 0, 1, 2 \rangle} \cdot \overbrace{\vec{r}\,'(t)}^{\vec{v}(t)}\, dt$$

$$= \int_0^{\pi} 1 \cdot (1 + \sin t) + 2(2 - t)\, dt$$

$$= \int_0^{\pi} \sin t + 5 - 2t\, dt = -\cos t + 5t - t^2 \Big|_0^{\pi}$$

$$= \boxed{2 + 5\pi - \pi^2 \text{ J}}$$

**4. (10 points)**

**a.** Consider the vectors $\langle 1, 2, -3\rangle$ and $\langle 4, 2, 1\rangle$. Is the angle between the two vectors acute (less than a right angle), obtuse (more than a right angle) or right? Justify briefly.

$$\langle 1,2,-3\rangle \cdot \langle 4,2,1\rangle = 4+4-3$$
$$= 5 > 0 \Rightarrow \text{acute}$$

**b.** Compute the area of the parallelogram generated by the vectors $\mathbf{v} = \langle 10, 20, 2\rangle$ and $\mathbf{w} = \langle -20, 10, 1\rangle$. The parallelogram has vertices $\mathbf{0}$, $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{v} + \mathbf{w}$.

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 10 & 20 & 2 \\ -20 & 10 & 1 \end{vmatrix} = \hat{i}(20-20) + \hat{j}(-40-10) + \hat{k}(100+400)$$
$$= -50\hat{j} + 500\hat{k}$$

$$\text{area} = \boxed{\sqrt{500^2 + 50^2}} \approx 502.5$$

**c.** Write down the formula of a plane parallel to the plane $2x - 3\cancel{x} + 5z = 9$ but that passes through the point $(1, 2, 1)$.

$y \leftarrow$ misprint

$$\boxed{2(x-1) - 3(y-2) + 5(z-1) = 0}$$

[as written $2(x-1) - 3(x-1) + 5(z-1) = 0$]

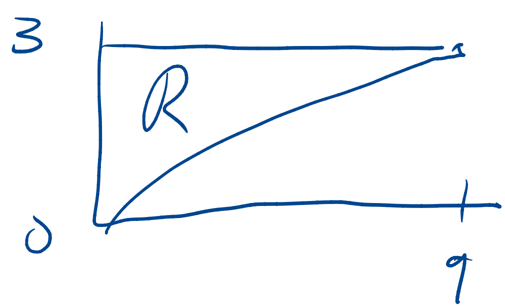
**5. (10 points)**

Consider the iterated integral

$$\int_0^9 \int_{\sqrt{x}}^3 \sqrt{y^3 + 1}\, dy\, dx.$$

This corresponds to a double integral over a region $\mathcal{R}$ in the plane

**a.** Sketch the region $\mathcal{R}$ in the area below. Label your axes with any interesting points.




$y = \sqrt{x}$

$y^2 = x$

**b.** Compute the integral by interchanging the order of integration.

$$\int_0^3 \int_0^{y^2} \sqrt{y^3+1}\, dx\, dy = \int_0^3 y^2\sqrt{y^3+1}\, dy$$

$$= \frac{2}{9}\left(y^3+1\right)^{3/2} \Big|_0^3$$

$$= \boxed{\frac{2}{9}\left[28^{3/2} - 1\right]}$$

**6. (10 points)**

Consider the vector field

$$\mathbf{F} = \left\langle 3x^2 - e^{y^2}, -2xye^{y^2} - \cos(y) \right\rangle$$

**a. 5 points** Show that the vector field is conservative by finding a potential for it.

$$\frac{\partial f}{\partial x} = 3x^2 - e^{y^2} \Rightarrow f(x,y) = x^3 - xe^{y^2} + h(y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = -2xye^{y^2} + h'(y)$$

but $\frac{\partial f}{\partial y} = -2xye^{y^2} - \cos(y)$ by above

$$\Rightarrow h'(y) = -\cos(y) \Rightarrow h(y) = -\sin(y) + c$$

$$\boxed{f(x,y) = x^3 - xe^{y^2} - \sin(y) + c}$$

**b. 5 points** Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $C$ is the curve parameterized by

$$\mathbf{r}(t) = \left\langle \sin(\pi t/2), 1 - t^2 \right\rangle$$

for $-1 \leq t \leq 1$. Keep in mind that you just showed that the vector field is conservative! But take care: this curve may not end at the same place it starts.

$$\vec{r}(-1) = \langle \sin(-\pi/2), 0 \rangle = \langle -1, 0 \rangle$$

$$\vec{r}(1) = \langle \sin(\pi/2), 0 \rangle = \langle 1, 0 \rangle$$

$$\int_C \vec{F} \cdot d\vec{r} = f(1,0) - f(-1,0) = -\cos(0) + \cos(0)$$

$$= \boxed{0}$$

**7. (10 points)**

**a. 6 points** Find all the critical points of

$$f(x,y) = x^3/3 - xy + y^2/2$$

$$\frac{\partial f}{\partial x} = x^2 - y$$

$$\frac{\partial f}{\partial y} = -x + y$$

$\vec{\nabla} f = 0 \Rightarrow y = x$

$y = x^2$

$\Rightarrow x^2 = x$

$\Rightarrow x = 0, 1$

$$\boxed{(0,0), \quad (1,1)}$$

**b. 4 points** For each critical point you just found, classify it as a local min/max or saddle.

$f_{xx} = 2x \qquad f_{xy} = -1 \qquad f_{yy} = 2$

$$\mathcal{D} = \begin{vmatrix} 2x & -1 \\ -1 & 2 \end{vmatrix} = 4x - 1$$

at $(0,0)$ $\mathcal{D} = -1 < 0 \Rightarrow$ saddle

at $(1,1)$ $\mathcal{D} = 4 > 0$, $f_{yy} > 0 \Rightarrow$ local min

**8. (10 points)**

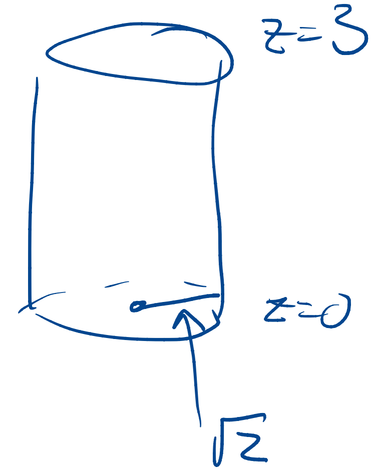
Consider the cylindrical region $\mathcal{E}$ with $x^2 + y^2 \leq 2$ and $0 \leq z \leq 3$. Let

$$\mathbf{F} = \left\langle x^3, y^3, z^2 \right\rangle.$$

**a. 5 points** Let $\mathcal{S}_1$ be the vertical wall of this cylinder (the part that looks like a tube). Set up an integral that computes the flux of $\mathbf{F}$ through $\mathcal{S}_1$ in the direction exterior to $\mathcal{E}$. Do NOT compute the integral. For full credit your answer should consist of an interated integral of two variables and a scalar field for the integrand.

$$\vec{r}(\theta, z) = \langle \sqrt{2}\cos\theta, \sqrt{2}\sin\theta, z \rangle$$

$$0 \leq \theta \leq 2\pi \qquad 0 \leq z \leq 3$$




$$\vec{r}_\theta = \langle -\sqrt{2}\sin\theta, \sqrt{2}\cos\theta, 0 \rangle$$

$$\vec{r}_z = \langle 0, 0, 1 \rangle$$

$$\vec{r}_\theta \times \vec{r}_z = \langle \sqrt{2}\cos\theta, \sqrt{2}\sin\theta, 0 \rangle$$

$$\vec{F}(\vec{r}(\theta, z)) = \langle (\sqrt{2}\cos\theta)^3, (\sqrt{2}\sin\theta)^3, z^2 \rangle$$

$$\vec{F}(\vec{r}(\theta, z)) \cdot \vec{r}_\theta \times \vec{r}_z = 4\cos^4\theta + 4\sin^4\theta$$

$$\int_0^{2\pi}\int_0^3 4\cos^4\theta + 4\sin^4\theta \, dz \, d\theta$$

**Continued....**

**Problem 8 continued....**

Recall we are working with the cylindrical region $\mathcal{E}$ with $x^2 + y^2 \leq 2$ and $0 \leq z \leq 3$ and that

$$\mathbf{F} = \left\langle x^3, y^3, z^2 \right\rangle.$$

**b. 5 points** Compute the outward flux through the entire boundary of $\mathcal{E}$ by using the Divergence Theorem. You must compute an honest number for your answer.

$$\vec{\nabla} \cdot \vec{F} = 3x^2 + 3y^2 + 2z = 3r^2 + 2z$$

cylindrical coords:

$$\int_{-\pi}^{\pi} \int_0^{\sqrt{2}} \int_0^3 (3r^2 + 2z)\, dz\, r\, dr\, d\theta$$

$$= \int_{-\pi}^{\pi} \int_0^{\sqrt{2}} 3r^2 z + z^2 \Big|_0^3 r\, dr\, d\theta$$

$$= 2\pi \int_0^{\sqrt{2}} 9r^3 + 9r\, dr$$

$$= 18\pi \left( \frac{r^4}{4} + \frac{r^2}{2} \Big|_0^{\sqrt{2}} \right) = 18\pi (1 + 1)$$

$$= \boxed{36\pi}$$

$$P = \frac{knT}{V}$$

**9. (10 points)**

Recall that the ideal gas law states

$$PV = knT$$

where $P$ is the pressure (in $\mathrm{N/m^2}$) of a gas, $V$ is the volume it occupies in $\mathrm{m^3}$, $T$ is the temperature in degrees Kelvin, $n$ is the number of molecules (counted in moles) and $k$ is the universal gas constant (approximately 8.314 J/mol/K).

Suppose 10 moles of gas occupies a 0.5 $\mathrm{m^3}$ region at a temperature $T = 270K$.

**a. 2 points** What is the resulting pressure?

$$P = \frac{knT}{V} = \frac{8{,}314 \cdot 10 \cdot 270}{0{,}5} \approx 45\,\mathrm{kPa}$$

**b. 8 points** Use differentials (or the linearization) to estimate the change in the pressure if the temperature is increased one degree Kelvin and the volume is decreased 0.01 $\mathrm{m^3}$. No credit will be awarded for an exact computation. You are welcome to leave your answer in a form so that a person with a calculator could compute the answer as a single number.

$$dP = \frac{\partial P}{\partial V}\,dV + \frac{\partial P}{\partial T}\,dT$$

$$= \frac{-knT}{V^2}\,dV + \frac{kn}{V}\,dT$$

$$= \frac{kn}{V}\left[-\frac{T}{V}\,dV + dT\right] = \frac{83{,}14}{0{,}5}\left[-\frac{270}{0.5}\cdot 0.01 + 1\right]$$

$$\approx -731\ \mathrm{Pa}$$

**10. (10 points)**

Consider the surface $\mathcal{S}$ given by the upside down paraboloid with $z = 2 - x^2 - y^2$ with $0 \le z \le 2$ oriented with its unit normal pointing in the positive $z$ direction. Let $\mathbf{Z} = \langle -y + z, x - z, xy \rangle$.

Stokes' Theorem says that

$$\iint_S \nabla \times \mathbf{Z} \cdot \mathbf{n}\, dS = \int_C \mathbf{Z} \cdot d\mathbf{r}$$

so long as $\mathbf{n}$ and $C$ are consistently oriented.

**a. 4 points** Write down $\nabla \times \mathbf{Z}$.

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -y+z & x-z & xy \end{vmatrix} = \hat{i}(x-1) + \hat{j}(1-y) + \hat{k}(1-1)$$

$$= (x-1)\hat{i} + (1-y)\hat{j}$$

**b. 6 points** Use Stokes' Theorem to determine the flux $\iint_S \nabla \times \mathbf{Z} \cdot \mathbf{n}\, dS$ by computing a **line integral** instead.

boundary: $z = 0$, $\quad 0 = 2 - x^2 - y^2$

$x^2 + y^2 = 2$

$\vec{r}(t) = \langle \sqrt{2}\cos t, \sqrt{2}\sin t, 0 \rangle$

$\vec{r}\,'(t) = \langle -\sqrt{2}\sin t, \sqrt{2}\cos t, 0 \rangle$

$\vec{Z}(\vec{r}(t)) = \langle -\sqrt{2}\sin t, \sqrt{2}\cos t, 2\sin t\cos t \rangle$

$$\int_0^{2\pi} \vec{Z} \cdot \vec{r}\,'(t)\, dt = \int_0^{2\pi} 2\sin^2 t + 2\cos^2 t \, dt$$

$$= \boxed{4\pi}$$

Extra credit (4 points):

Suppose **F** is a conservative vector field. Explain why the equation from Stoke's Theorem reduces to $0 = 0$. Partial credit will be awarded for correctly explaining why each side of this equation becomes a zero.

1) For a conservative vector field

$$\vec{\nabla} \times \vec{F} = 0 \Rightarrow \iint_{\mathcal{S}} (\vec{\nabla} \times \vec{F}) \cdot \vec{n} \, dS = 0$$

2) For a conservative vector field, and $C$ a curve from $p$ to $q$,

$$\int_C \vec{\nabla} f \cdot d\vec{r} = f(q) - f(p).$$

In particular, for a loop, $p = q$ so $\int_C \vec{\nabla} f \cdot d\vec{r} = 0$.

That is, if $\vec{F}$ is conservative, $\int_C \vec{F} \cdot d\vec{r} = 0$