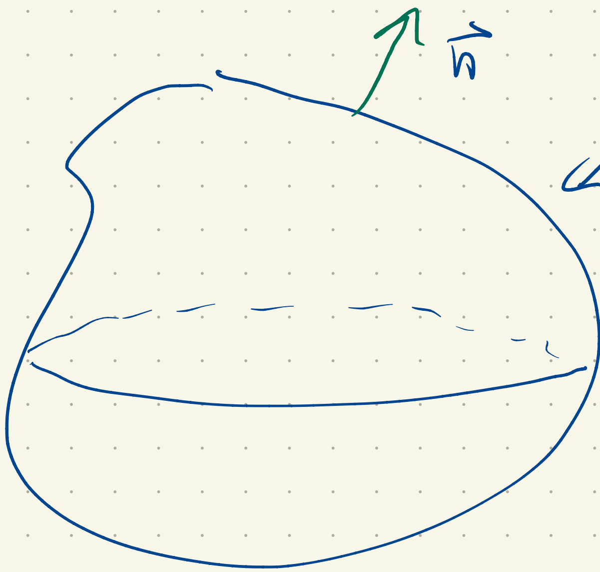
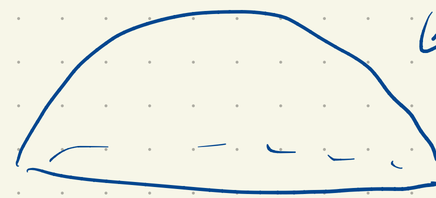
One more cousin of FTC

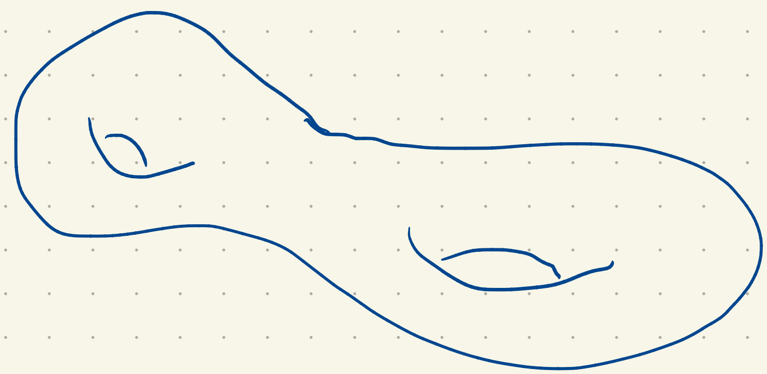
Divergence Theorem



S encloses a region

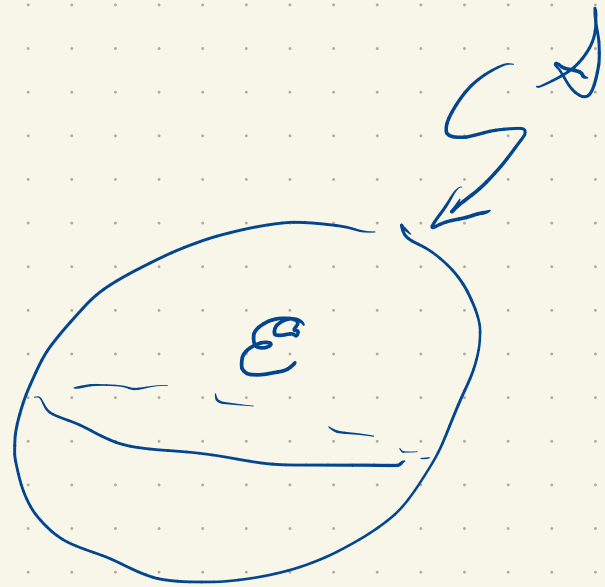
does not





Outward unit normal vector $\vec{n}$

$$\iint_{\mathcal{S}} \vec{X}\cdot\vec{n}\,dS = \iiint_{\mathcal{E}} \operatorname{div}\vec{X}\,dV$$



Any vector field $\vec{X}$.

$$f(b)-f(a) = \int_a^b f'(x)\,dx$$

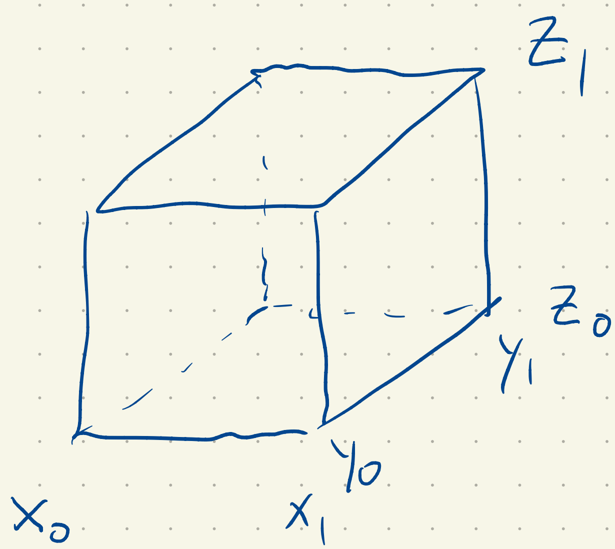
$$\int_{\mathcal{C}} \vec{V}\cdot d\vec{r} = \int_{\mathcal{S}} (\vec{\nabla}\times\vec{V})\cdot\vec{n}\,dS$$

Fluid density ρ and velocity $\vec{v}$

$$\vec{U} = \rho \vec{v} \qquad \iint_{\mathcal{S}} \vec{U} \cdot \vec{n} \, dS$$

rate at which mass is leaving the region

interior region.



$$\vec{X} = \langle P, Q, R \rangle$$

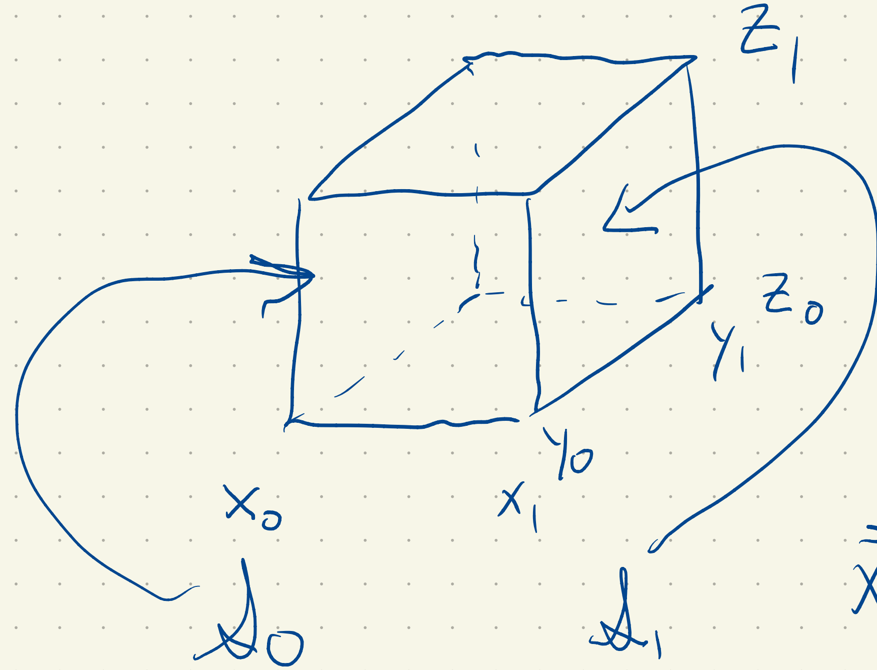
$$\vec{\nabla} \vec{X} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$\iiint_{\mathcal{E}} \frac{\partial P}{\partial x}\, dV = \int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} \frac{\partial P}{\partial x}\, dz\, dy\, dx$$

$$= \int_{y_0}^{y_1}\int_{z_0}^{z_1}\int_{x_0}^{x_1} \frac{\partial P}{\partial x}\, dx\, dz\, dy$$

$$= \int_{y_0}^{y_1}\int_{z_0}^{z_1} P(x_1,y,z) - P(x_0,y,z)\, dz\, dy$$

$$= \int_{y_0}^{y_1}\int_{z_0}^{z_1} P(x_1,y,z)\, dz\, dy - \int_{y_0}^{y_1}\int_{z_0}^{z_1} P(x_0,y,z)\, dz\, dy$$



$\vec{X} = \langle P, Q, R \rangle$

$\hat{i} = \langle 1, 0, 0 \rangle$

$$\iint_{S_1} P \, dA - \iint_{S_0} P \, dA$$

On surface S_1 $\vec{n} = \hat{i}$, $\vec{X} \cdot \vec{n} = \vec{X} \cdot \hat{i} = P$

S_0 $\vec{n} = -\hat{i}$ $\vec{X} \cdot \vec{n} = \vec{X} \cdot (-\hat{i}) = -P$

$$\iint_{S_1} \vec{X}\cdot\vec{n}\,dS + \iint_{S_0} \vec{X}\cdot\vec{n}\,dS$$

$$\iiint_{E} \frac{\partial P}{\partial x}\,dV = \iint_{S_1} \vec{X}\cdot\vec{n}\,dS + \iint_{S_0} \vec{X}\cdot\vec{n}\,dS$$

$$\iiint_{E} \underbrace{\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)}_{\vec{\nabla}\cdot\vec{X}} dV = \checkmark + 2\text{ more} + 2\text{ more}$$

$$= \iint_{S} \vec{X}\cdot\vec{n}\,dS$$

$\vec{X}$

$$\iiint_{E} \operatorname{div} \vec{X} \, dV = \iint_{S} \vec{X} \cdot \vec{n} \, dS$$

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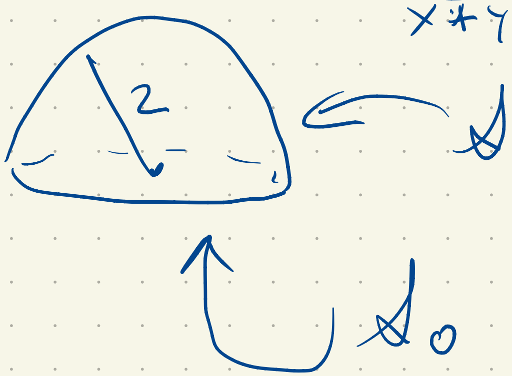
e.g. $\vec{X} = x\hat{i} + y\hat{j} + z\hat{k}$

$x^2 + y^2 + z^2 = 4, \quad z \geq 0$



$z = 0$

$\vec{n} = -\hat{k}$

$\vec{X} \cdot \vec{n} = -z = 0$

$$\iiint_{\mathcal{E}} \vec{\nabla}\cdot\vec{x}\, dV, \qquad \iint_{S} \vec{x}\cdot\vec{n}\, dS = \iint_{S_0} \vec{x}\cdot\vec{n}\, dS + \iint_{S_1} \vec{x}\cdot\vec{n}\, dS$$

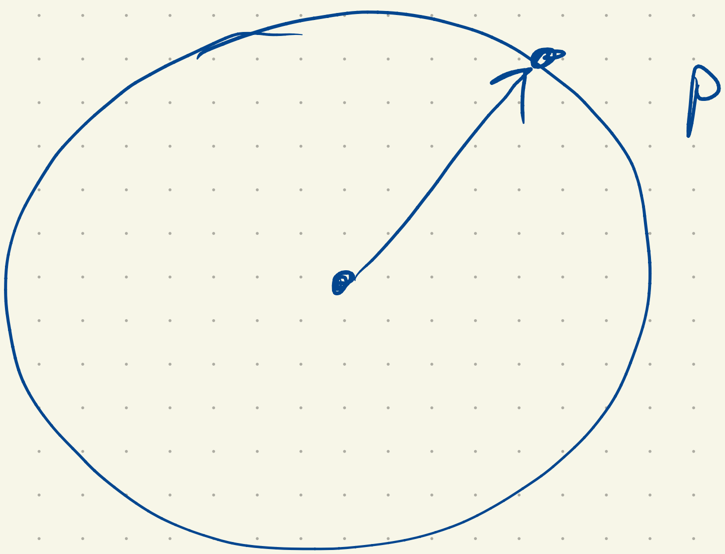
$$\vec{\nabla}\cdot\vec{x} = 3$$

$$\iiint_{\mathcal{E}} \vec{\nabla}\cdot\vec{x}\, dV = \iiint_{\mathcal{E}} 3\, dV = 3 \iiint_{\mathcal{E}} dV = 3\cdot\frac{1}{2}\,\frac{4}{3}\pi\cdot 2^3$$

$$= 16\pi$$

$$\iint_{S_0} \vec{x}\cdot\vec{n}\, dS = \iint_{S_0} 0\, dS = 0$$

$$\iint_{S_1} \vec{X}\cdot\vec{n}\, dS$$



$\langle x, y, z\rangle$ is parallel to $\vec{n}$

on S_1

has length 2.

$\vec{n} = \frac{1}{2}\langle x, y, z\rangle$ on S_1

$\vec{X} = \langle x, y, z\rangle$

$$\vec{x}\cdot\vec{n} = \frac{1}{2}\left(x^2+y^2+z^2\right) = \frac{4}{2} = 2$$

$$\iint_{S_1} \vec{x}\cdot\vec{n}\, dS = \iint_{S_1} 2\, dS = 2\iint_{S_1} dS = 2\cdot\frac{1}{2}\, 4\pi(2)^2$$

$$= 4\pi\cdot 4$$

$$= 16\pi$$

$$\overset{\text{volume int}}{16\pi} = \overset{\text{int over } S_0}{0} + \overset{S_1}{16\pi}$$