Flux integrals $\iint_{S} \vec{X}\cdot\vec{n}\, dS$

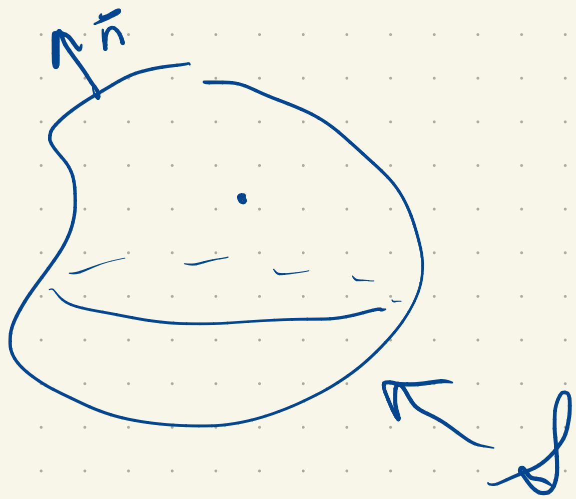
If $\vec{X} = \underset{\substack{\uparrow \\ \text{mass density} \\ \text{(fluids)}}}{\rho}\,\underset{\substack{\uparrow \\ \text{velocity}}}{\vec{v}}$ flux integral is mass / time going through S.

$\rho \rightarrow$ charge density (Coulombs / volume)

$\vec{X} = \rho\vec{v} = \vec{J}$ flux integral is charge per time flowing through S.

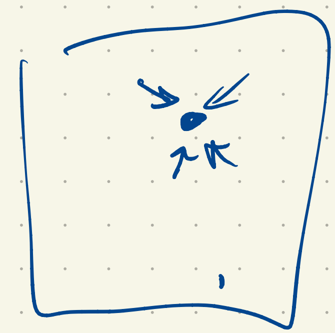
$\hookrightarrow$ current density

$T \to$ temperature $\qquad \vec{\nabla} T \qquad [\vec{\nabla} T] = K/m$



$k \to$ thermal conductivity

$J/_{Kms}$

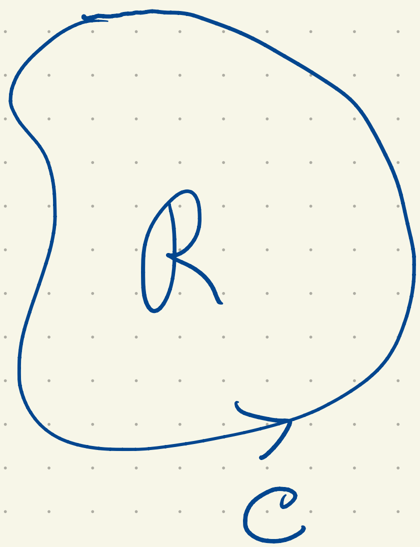


$$\iint_S (-k\vec{\nabla} T)\cdot\vec{n}\, dS \longrightarrow \text{Joules}/s$$

Energy (heat) passing through S to the exterior

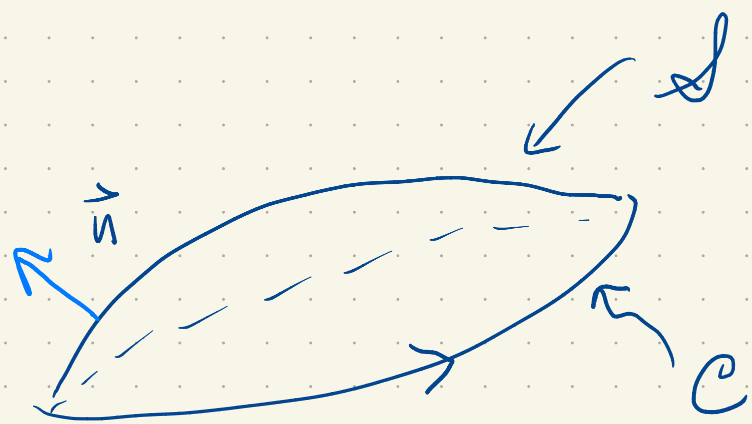
$$\iint_S \vec{E}\cdot\vec{n}\, dS \qquad \iint_S \vec{B}\cdot\vec{n}\, dS$$

Stokes' Theorem
Divergence Theorem] flavors of FTC



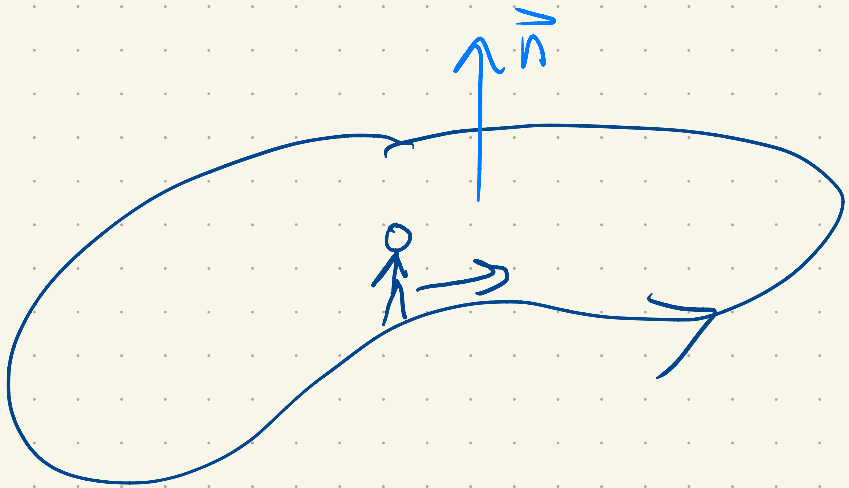
$$\vec{F} = \langle P, Q \rangle$$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_R (-P_y + Q_x)\, dA(x,y)$$



$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, dS$$

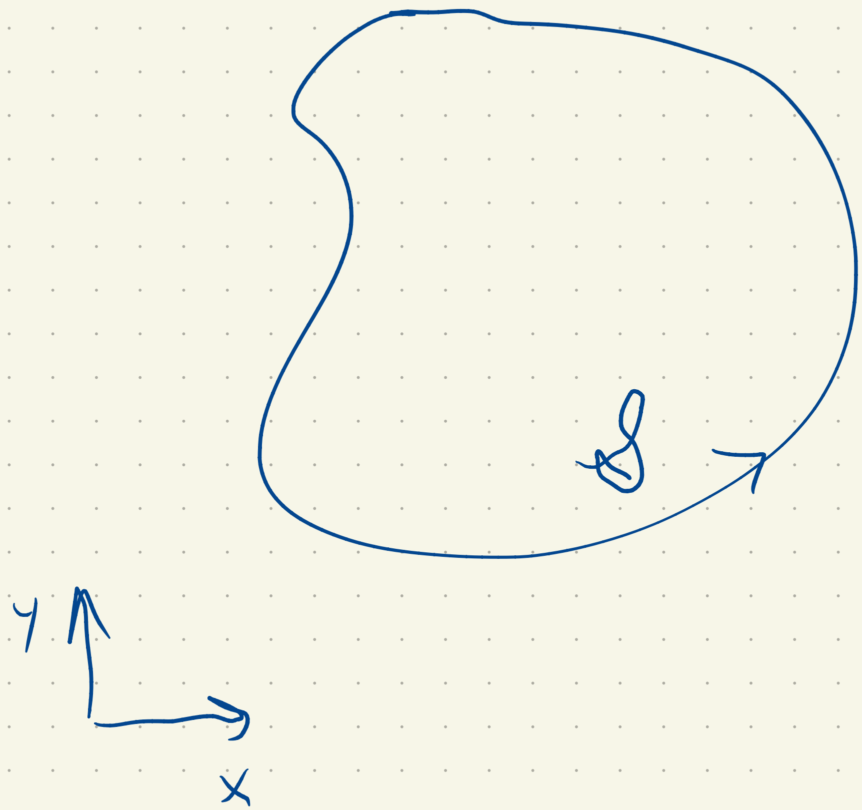
$$\int_a^b f'(x)\,dx = f(b) - f(a)$$



My head points in the normal direction and the surface is on my left as I walk around.

Suppose $F = \langle P, Q, 0 \rangle$ and that P and Q don't depend on z,

and that S is contained in x-y plane.



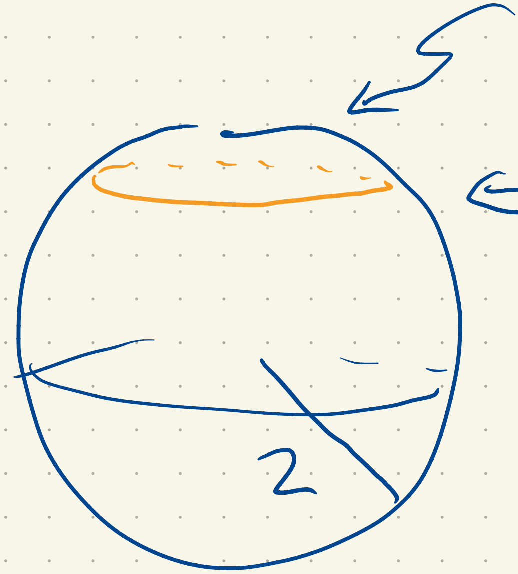
$$\vec{\nabla} \times \vec{F} = ?$$

$$\begin{vmatrix} \cdot & \cdot & \cdot \\ \partial_x & \partial_y & \partial_z \\ P & Q & 0 \end{vmatrix} = \langle 0, 0, Q_x - P_y \rangle$$

$$\vec{n} = \langle 0, 0, 1 \rangle$$

$$(\vec{\nabla} \times \vec{F}) \cdot \vec{n} = Q_x - P_y$$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S Q_x - P_y \, dS = \iint_S Q_x - P_y \, dA$$



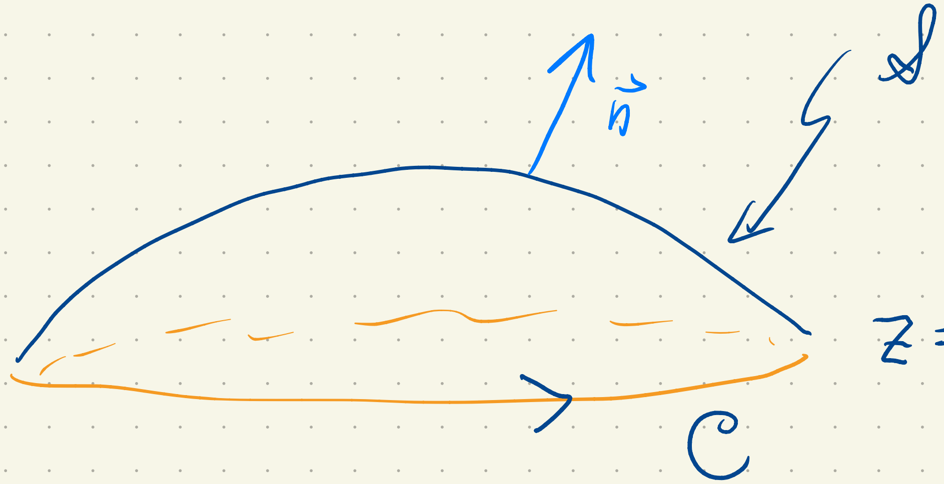
$x^2+y^2+z^2=4$

$z=\sqrt{3}$

$x^2+y^2+3=4$

$\underbrace{x^2+y^2=1}$

$\hookrightarrow$ boundary, $z=\sqrt{3}$



$z=\sqrt{3}$

$$\vec{F}=\langle xz,\ yz+x,\ xy\rangle$$

We will show

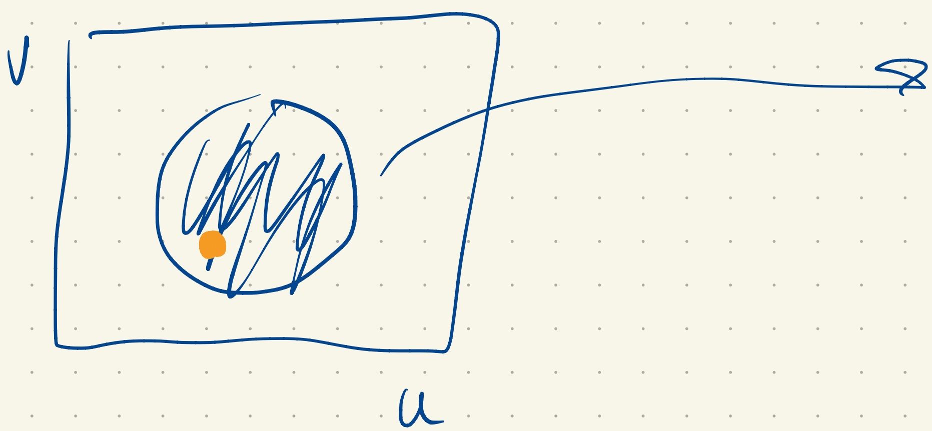
$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,dS$$

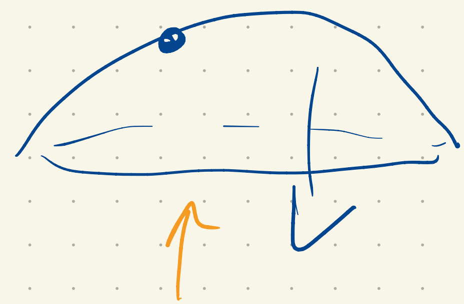
$\partial_x \quad \partial_y \quad \partial_z \quad \rightarrow \quad \langle x-y,\ x-y,\ 1 \rangle = \vec{\nabla} \times \vec{F}$

$xz \quad (yz+x) \quad xy$

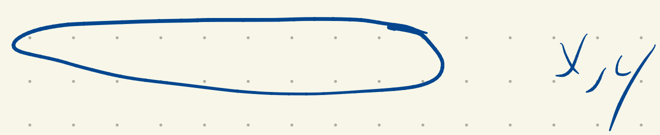
$$\vec{r}(u,v) = \langle u,\ v,\ \sqrt{4-u^2-v^2} \rangle$$

$$u^2+v^2 \leq 1$$





$\langle u,\ v,\ \sqrt{4-u^2-v^2} \rangle$



$$\vec{r}_u = \left\langle 1, 0, \frac{-u}{\sqrt{4-u^2-v^2}} \right\rangle$$

$$\vec{r}_v = \left\langle 0, 1, \frac{-v}{\sqrt{4-u^2-v^2}} \right\rangle$$

$$\vec{r}_u \times \vec{r}_v = \left\langle \frac{u}{\sqrt{4-u^2-v^2}}, \frac{v}{\sqrt{4-u^2-v^2}}, 1 \right\rangle$$

$$\vec{X} \cdot \vec{n}\, dS \longrightarrow \vec{X}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dv$$

$$\vec{\nabla} \times \vec{F} = \langle u-v, u-v, 1 \rangle$$

$$(\vec{\nabla} \times \vec{F}) \cdot (\vec{r}_u \times \vec{r}_v) = \frac{u^2-uv}{\sqrt{4-u^2-v^2}} + \frac{uv-v^2}{\sqrt{4-u^2-v^2}} + 1$$

$$= \frac{u^2 - v^2}{\sqrt{4-u^2-v^2}} + 1$$

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n} \, dS = \iint_D \left(\frac{u^2 - v^2}{\sqrt{4-u^2-v^2}} + 1 \right) du \, dv$$

$D \hookrightarrow u^2 + v^2 \leq 1$

$$\int_{-\pi}^{\pi} \int_0^1 \left[\frac{(r\cos\theta)^2 - (r\sin\theta)^2}{\sqrt{4-r^2}} + 1 \right] r \, dr \, d\theta$$

$$\int_{-\pi}^{\pi} \cos^2\theta \, d\theta = \pi \qquad \int_{-\pi}^{\pi} \frac{r^2\cos^2\theta}{\sqrt{4-r^2}} \, r \, d\theta = \pi \frac{r^3}{\sqrt{4-r^2}}$$

$$\int_{-\pi}^{\pi} \sin^2\theta \, d\theta = \pi \qquad \int_{-\pi}^{\pi} \frac{r^2\sin^2\theta}{\sqrt{4-r^2}} \, r \, d\theta = \pi \frac{r^3}{\sqrt{4-r^2}}$$

$$\int_{-\pi}^{\pi}\int_0^1 r \, dr \, d\theta = \int_{-\pi}^{\pi} \frac{1}{2} \, d\theta = \boxed{\pi}$$

$$\vec{\sigma}(t) = \langle \cos(t), \sin(t), \sqrt{3} \rangle$$

$$\vec{F} = \langle xz, \; yz + x, \; xy \rangle$$

$$= \langle \sqrt{3}\cos(t), \sqrt{3}\sin(t) + \cos(t), \cos(t)\sin(t) \rangle$$

$$\vec{\sigma}'(t) = \langle -\sin(t), \cos(t), 0 \rangle$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\sigma(t)) \cdot \vec{\sigma}'(t)\, dt$$

$$= \int_0^{2\pi} \langle \sqrt{3}\cos(t), \sqrt{3}\sin(t) + \cos(t), \cos(t)\sin(t) \rangle \cdot \langle -\sin(t), \cos(t), 0 \rangle\, dt$$

$$= \int_0^{2\pi} -\sqrt{3}\cos(t)\sin t + \sqrt{3}\sin(t)\cos(t) + \cos^2(t)\, dt$$

$$= \int_0^{2\pi} \cos^2(t)\,dt$$

$$= \boxed{\pi}$$