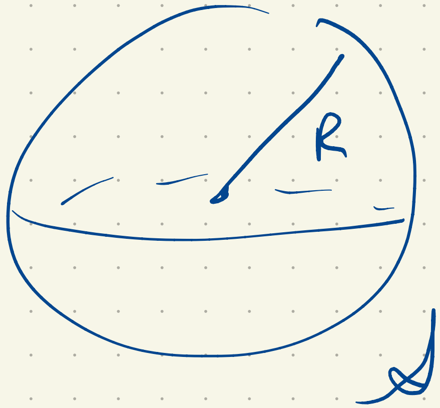
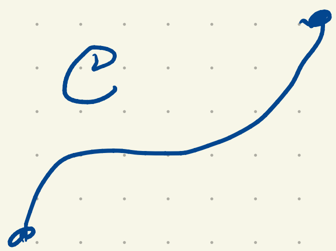


$f(x,y,z)$

$$\iint_{S} f(x,y,z)\,dS$$

$\vec{r}(u,v)$

$$\boxed{\iint_{D} f(\vec{r}(u,v))\,\|\vec{r}_u \times \vec{r}_v\|\,du\,dv}$$



$\rho(x,y,z)$

$$\int_{C} \rho(x,y,z)\,ds$$

$\vec{r}(t) \qquad a \le t \le b$

$$\boxed{\int_a^b \rho(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt}$$

$$\iint_{S} z^2 \, dS$$

$$dS = R^2 \cos v \, du \, dv$$

$$dx\,dy = r\,dr\,d\theta$$

$$\int_{-\frac{\pi}{2}}^{\pi/2} \int_{-\pi}^{\pi} R^2 \sin^2 v \quad R^2 \cos v \, du \, dv$$

$$= R^4 \int_{-\frac{\pi}{2}}^{\pi/2} \int_{-\pi}^{\pi} \sin^2 v \cos v \, du \, dv = R^4 \frac{4}{3}\pi$$

$$\vec{r}(u,v) = \langle R \cos u \cos v, R \sin u \cos v, R \sin v \rangle$$

Flux

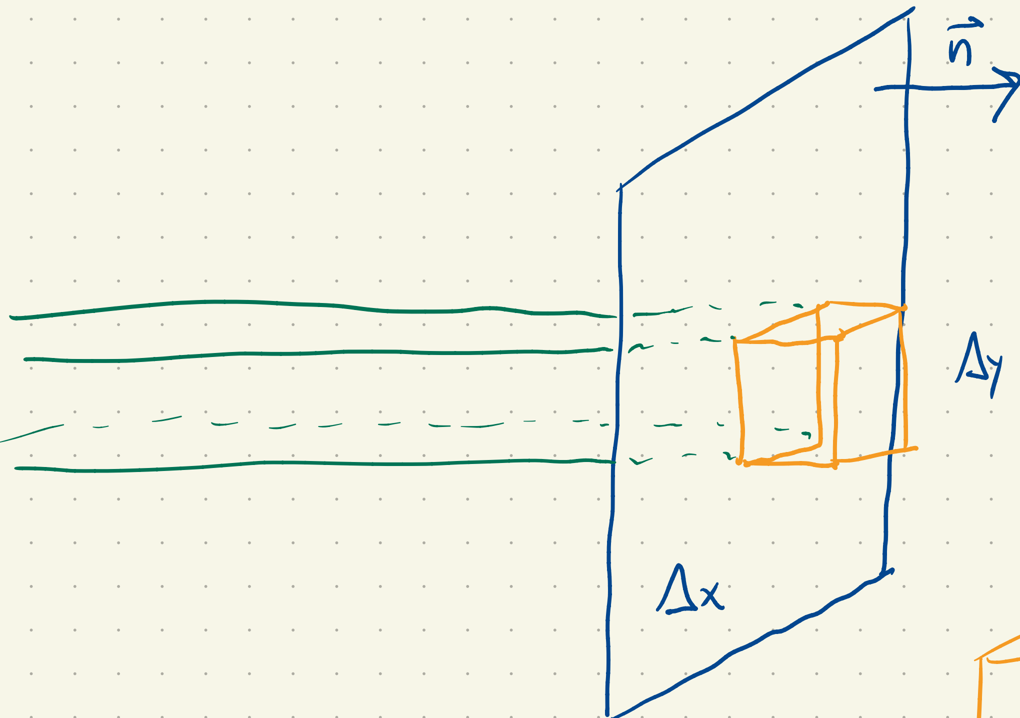
fluid density ρ (mass/volume)

velocity $\vec{v}$ (length/time)

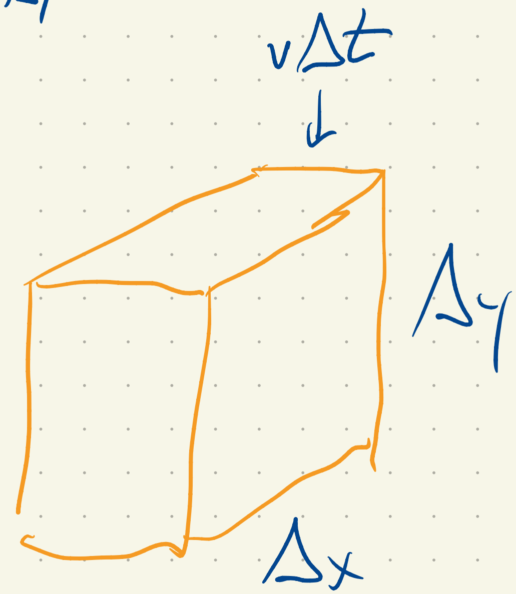
$\rho\vec{v}$ how mass is transported

"mass flux density"

$\vec{n}$ unit normal vector



what is the rate at which mass is being transported through the little square?



Δt

$\vec{v} = v\vec{n}$

Mass that went through in time interval Δt

$$\rho \, v \, \Delta x \, \Delta y \, \Delta t$$

So the rate (mass/time) is $\rho \, v \, \Delta x \, \Delta y$

$$\frac{\text{mass}}{\text{length}^3} \; \frac{\text{length}}{\text{time}} \; \text{length}^2 = \frac{\text{mass}}{\text{time}}$$

If $\vec{v}$ is parallel to the plane then no flux through the plane.

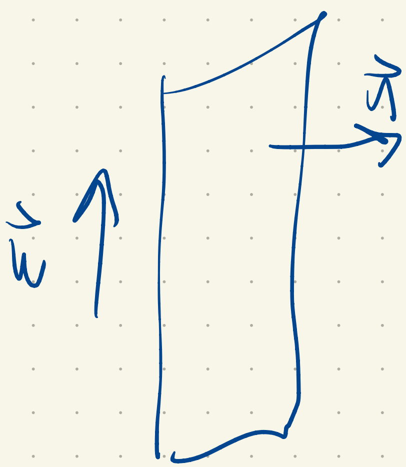
If $\vec{v} = \vec{w} + c\,\vec{n}$ where $\vec{w}$ is parallel to the surface

Only the term $c\vec{n}$ contributes to the flux

$$\vec{v}\cdot\vec{n} = (\vec{w}+c\vec{n})\cdot\vec{n} = \vec{w}\cdot\vec{n} + c\,\underbrace{\vec{n}\cdot\vec{n}}$$

$$= 0 + c\cdot 1$$

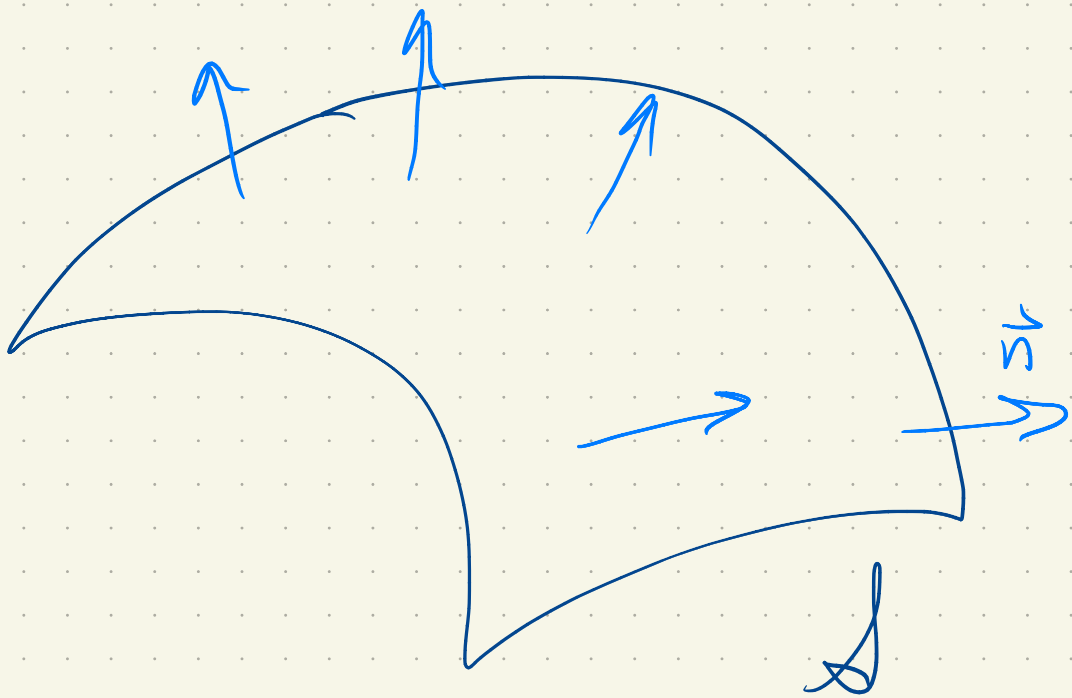
$$= c$$



Now the flux through the little square

is

$$\rho\,\vec{v}\cdot\vec{n}\,\underbrace{\Delta x\,\Delta y}_{\text{little bit of surface area}}$$



$\rho \qquad \vec{v}$

mass flux (mass/time) $\qquad \displaystyle\iint_S \rho\vec{v}\cdot\vec{n}\,dS$

$\vec{X} \qquad \rho\vec{v}$

In general if $\vec{X}$ is a vector field

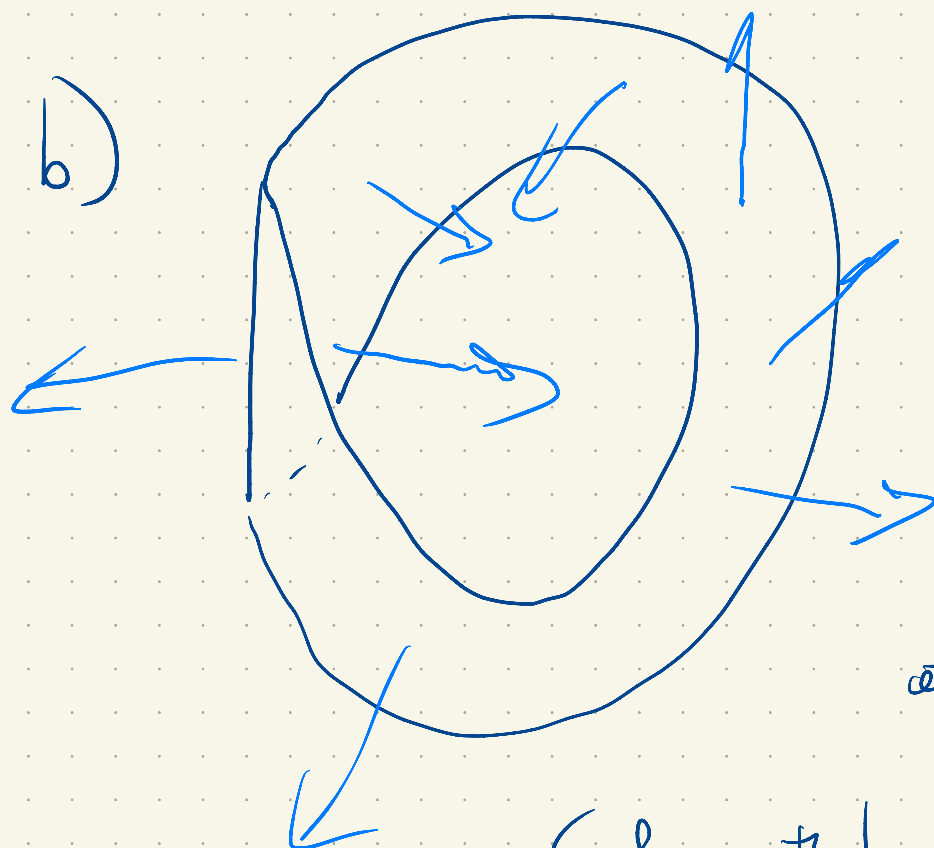
we call $\displaystyle\iint_S \vec{X}\cdot\vec{n}\,dS$ the flux of $\vec{X}$ through S.

Caveats

a) orientation matters

There is a choice of unit normal vector and the two differ by sign.

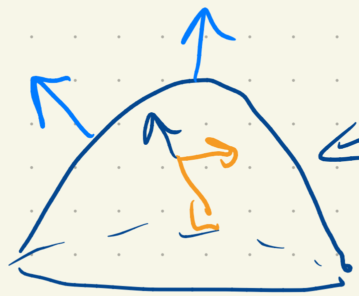
b)



Some surfaces (Möbius strip!) don't have a consistent choice of unit normal.

Surfaces that do are called orientable.

$\mathcal{S}$



$1-x^2-y^2 = z$

$z = 0$

$\vec{X} = \langle y, x, z \rangle$

Want to compute $\iint\limits_{\mathcal{S}} \vec{X} \cdot \vec{n} \, dS =$

$\vec{r}(x,y) = \langle x, y, 1-x^2-y^2 \rangle$ $\quad x^2+y^2 \le 1$

We need $\vec{r}_x \times \vec{r}_y$ for two reasons!

$$dS = \|\vec{r}_x \times \vec{r}_y\| \, dx \, dy$$

$\vec{r}_x \times \vec{r}_y$ is normal to the surface!

$$\vec{n} = \frac{\vec{r}_x \times \vec{r}_y}{\|\vec{r}_x \times \vec{r}_y\|}$$

$$\vec{X} \cdot \vec{n} \, dS = \vec{X} \cdot \frac{\vec{r}_x \times \vec{r}_y}{\|\vec{r}_x \times \vec{r}_y\|} \|\vec{r}_x \times \vec{r}_y\| \, dx \, dy$$

$$= \vec{X} \cdot (\vec{r}_x \times \vec{r}_y) \, dx \, dy$$

$$\vec{r}(x,y) = \langle x, y, 1-x^2-y^2 \rangle$$

$$\vec{r}_x = \langle 1, 0, -2x \rangle$$

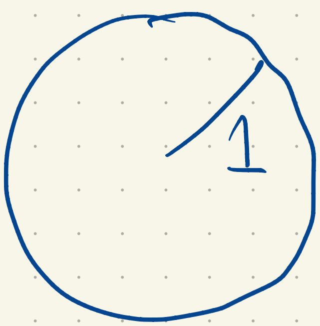
$$\vec{r}_y = \langle 0, 1, -2y \rangle$$

$$\vec{r}_x \times \vec{r}_y = \langle 2x, 2y, 1 \rangle$$

$$\vec{X} = \langle y, x, z \rangle = \langle y, x, 1-x^2-y^2 \rangle$$

$$\vec{X} \cdot (\vec{r}_x \times \vec{r}_y) = 2xy + 2xy + (1-x^2-y^2)$$

$$\iint_{S} \vec{X}\cdot\vec{n}\, dS = \iint_{D_{u,v}} \vec{X}(\vec{r}(x,y))\cdot(\vec{r}_x \times \vec{r}_y)\, dx\, dy$$



$$= \iint_{D} 4xy + 1 - x^2 - y^2 \, dx\, dy$$

$$= \int_{-\pi}^{\pi}\int_{0}^{1} (4r^2\cos\theta\sin\theta + 1 - r^2)\, r\, dr\, d\theta$$

$$= \frac{\pi}{2}$$