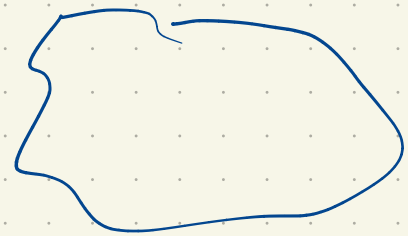
$$\underbrace{\partial \quad \vec{V}} \longrightarrow$$



$$\vec{V} = \langle P, Q, R \rangle$$

$$\vec{\nabla} \cdot \vec{V} = \partial_x P + \partial_y Q + \partial_z R$$

Curl curl $\vec{V}$ $\quad \vec{\nabla} \times \vec{V}$

How to compute?

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix} = (R_y - Q_z)\hat{i} - (R_x - P_z)\hat{j} + (Q_x - P_y)\hat{k}$$

$$\vec{V} = \langle P, Q \rangle$$

$$- \frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}$$

$$\vec{V} = \langle xz, \; xy^2z, \; -e^{2y} \rangle$$

$$\vec{\nabla} \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ xz & xy^2z & -e^{2y} \end{vmatrix} = \left(-2e^{2y} - xy^2\right)\hat{i} - \left(-x\right)\hat{j}$$

$$+ \left(y^2z\right)\hat{k}$$

Two identies: $\quad f$

$$\vec{\nabla} \times (\vec{\nabla} f) = \vec{0}$$

$$\vec{\nabla} f = \langle f_x, f_y, f_z \rangle$$

$$\vec{\nabla} \times \vec{\nabla} f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{vmatrix} = (f_{zy} - f_{yz})\hat{i}$$

$$- (f_{zx} - f_{xz})\hat{j}$$

$$(f_{yx} - f_{xy})\hat{k} = \vec{0}$$

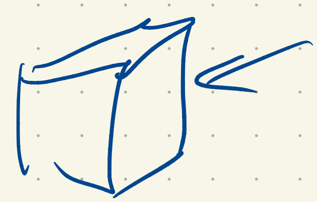
$$\vec{W} = \vec{\nabla} f$$

$\hookrightarrow$ "a potential"

If $\vec{\nabla} \times \vec{W} \neq 0$ then $\vec{W}$ is not conservative.

Conversely if $\vec{\nabla} \times \vec{W} = 0$ and the domain of $\vec{W}$ is simply connected (no hole) then $\vec{V}$ is conservative

(boxes and balls are always ok)



$$\vec{\nabla} \cdot (\underbrace{\vec{\nabla} \times \vec{V}}_{\partial P \ \partial Q \ \partial R}) = 0 \qquad \partial^2 P \quad \partial^2 Q \quad \partial^2 R$$

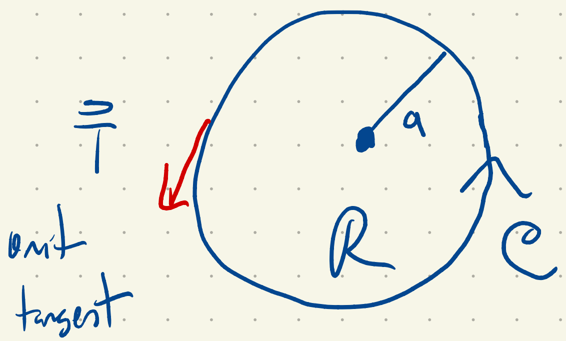
If $\vec{W}$ satisfies $\vec{\nabla} \cdot \vec{W} = 0$ then on boxes or balls there exists a vector field $\vec{V}$ where $\vec{W} = \vec{\nabla} \times \vec{V}$

$$\vec{\nabla} \cdot (\vec{\nabla} f) = \Delta f = \partial_x^2 f + \partial_y^2 f + \partial_z^2 f$$

Lapacian

$\Delta f = 0$ is rare and special ("harmonic")

$\vec{V} = \langle P, Q \rangle$ (velocity)



$$\frac{1}{2\pi a}\oint_C \vec{V}\cdot d\vec{r} = \underbrace{\frac{1}{2\pi a}\oint_C \underbrace{\vec{V}\cdot\vec{T}}_{\text{amount of } \vec{V} \text{ tangential to the circle (with sign)}}\, ds}_{\text{average velocity around the circle}}$$

distance around is $2\pi a$

$$\frac{1}{(2\pi a)^2}\oint_C \vec{V}\cdot d\vec{r} = \frac{1}{\text{time to go around the circle}}$$

rotations per time

kind of angular velocity

$$\frac{1}{(2\pi a)^2}\oint_C \vec{V}\cdot d\vec{r} = \frac{1}{(2\pi a)^2}\iint_R -P_y + Q_x \, dx\,dy$$

$$= \frac{1}{4\pi}\underbrace{\left[\frac{1}{\pi a^2}\iint_R \left(-P_y + Q_x\right)dx\,dy\right]}_{\text{average value of } (-P_y+Q_x) \text{ over the circle}}$$

$$\text{area}(R) = \pi a^2$$

$\frac{1}{4\pi}\left(-P_y + Q_x\right)$ is the angular velocity in cycles/time

over an infinitesmally small circle.

$\theta \uparrow$

$\frac{1}{2}\left(-P_y + Q_x\right)$ is angular velocity in radians/time

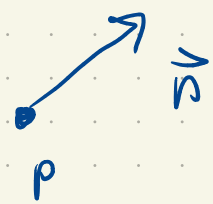
over an infinitesmaly small circle

"circulation"

curl $\vec{V}$ has a job:

pick a location p

pick a unit normal vector at p, call it $\vec{n}$

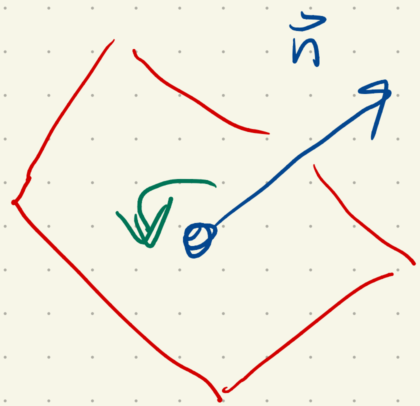


$\frac{1}{2}(\vec{\nabla} \times \vec{V}) \cdot \vec{n}$ is the circulation (rad/time)

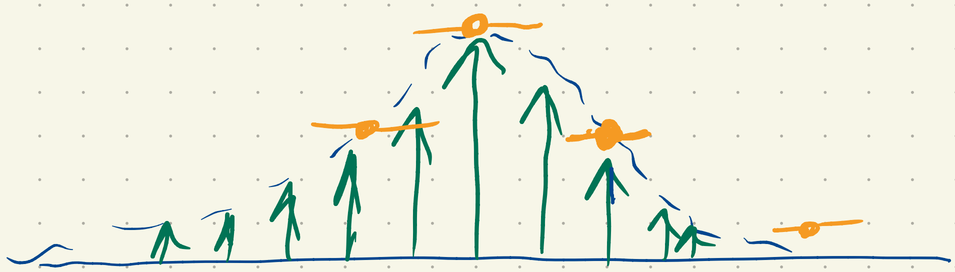
of the fluid in the plane

perpendicular to $\vec{n}$

as seen from $\vec{n}$.



$$\vec{V} = e^{-x^2}\,\hat{j}$$



$$\vec{\nabla} \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 0 & e^{-x^2} & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} - 2x e^{-x^2}\,\hat{k}$$

$$= -2x e^{-x^2}\,\hat{k}$$

$$(\vec{\nabla} \times \vec{V}) \cdot \hat{k} = -2x e^{-x^2}$$