Conservative Vector Fields

$\vec{F} \to$ vector field

We say that $\vec{F}$ is conservative if there exists a function f

such that $\vec{F} = \vec{\nabla} f$

in which case we call f a potential of $\vec{F}$.

Suppose $\vec{F} = \langle P, Q \rangle$ is conservative.

$P = \partial_x f$

$Q = \partial_y f$

Then $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

$$\partial_y P = \partial_y \partial_x f$$
$$= \partial_x \partial_y f$$
$$= \partial_x Q$$

$\vec{F} = \langle \underset{P}{x^2}, \underset{Q}{xy} \rangle$

$$\frac{\partial P}{\partial y} = 0 \qquad \frac{\partial Q}{\partial x} = y \qquad 0 \neq y \text{ generally}$$

$\Rightarrow \vec{F}$ is not conservative

Most vector fields are not conservative.

Many fundamental force fields in physics are conservative

($\vec{E}$ electric field, $\vec{G}$ gravitational field)

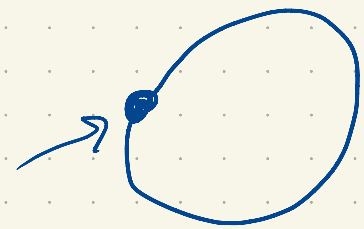
If $\vec{F}$ is conservative with potential f

$$\int_C \vec{F} \cdot d\vec{r} = f(q) - f(p)$$

3 properties of conservative vector fields

a) $\int_C \vec{F} \cdot d\vec{r}$ depends only on the endpoints of C and nothing else about C

b) $\oint_C \vec{F} \cdot d\vec{r} = 0$ for all loops C



c) "Mixed partial" property $\quad \vec{F} = \langle P, Q, R \rangle$

$P_y = Q_x, \quad P_z = R_x, \quad Q_z = R_y$

Question: If a vector field satisfies a) or b) or c) is it conservative?

In fact $\quad b) \Rightarrow a) \Rightarrow$ vector field is conservative (see text)

How about c)? $\quad \vec{F} = \langle P, Q \rangle$

If we know $P_y = Q_x$ can we conclude $\vec{F}$ is conservative an indeed can we find a potential?

$$\vec{F}(x,y) = \langle \underbrace{y^2 - 2x}_{P}, \underbrace{2xy}_{Q} \rangle$$

$$P_y \stackrel{?}{=} Q_x \qquad P_y = \underbrace{2y} \qquad Q_x = \underbrace{2y}$$

$$\longleftarrow = \longrightarrow$$

Can we find a potential? $\vec{F} = \vec{\nabla} f = \langle \partial_x f, \partial_y f \rangle$

$$\partial_x f = y^2 - 2x$$

$$\partial_y f = 2xy$$

If $\partial_x f = y^2 - 2x$ then $f(x,y) = xy^2 - x^2 + h(y)$

$$\partial_x f = y^2 - 2x + 0$$

$$f(x,y) = xy^2 - x^2 + h(y)$$

$$\left.\begin{array}{l} \text{want } \partial_y f = 2xy \\ \text{have } \partial_y f = 2xy + 0 + h'(y) \end{array}\right] \Rightarrow h'(y) = 0$$

$f(x,y) = xy^2 - x^2$ $\quad \partial_x f = y^2 - 2x$

$\partial_y f = 2xy$

$$\vec{F} = \langle \underbrace{3+2xy}_{P}, \underbrace{x^2-3y^2}_{Q} \rangle$$

Is there a potential?

$$\partial_y P = 2x$$

$2x = 2x$, so maybe it is conservative.

$$\partial_x Q = 2x$$

$$\partial_x f = 3+2xy$$

$$\partial_y f = x^2-3y^2$$

$$\partial_x f = 3+2xy \Rightarrow f(x,y) = 3x + x^2y + h(y)$$

Want $\partial_y f = x^2-3y^2$

Have $\partial_y f = 0 + x^2 + h'(y)$

$$\Rightarrow h'(y) = -3y^2$$

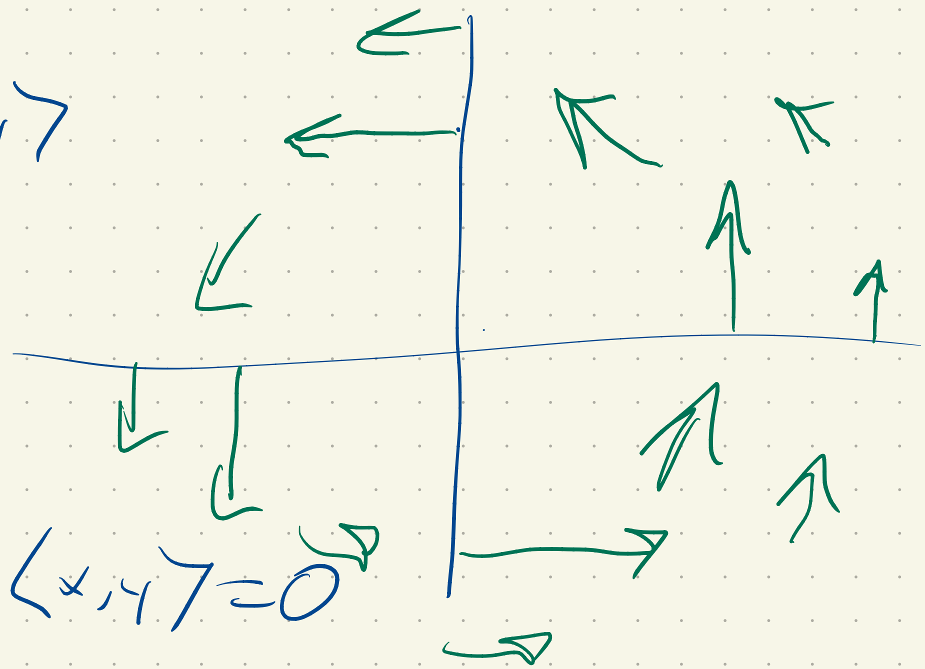
$$h(y) = -y^3 + c$$

$$f(x,y) = 3x + x^2y - y^3$$

It is almost true that if $P_y = Q_x$ then $\vec{F}$ is conservative.

$$\vec{F} = \langle P, Q \rangle \qquad P = \frac{-y}{x^2+y^2} \qquad Q = \frac{x}{x^2+y^2}$$

$\langle x, y \rangle$



$\langle -y, x \rangle \cdot \langle x, y \rangle = 0$

$$\int_C \vec{F} \cdot d\vec{r}$$