Last class

$\vec{u} = \langle u_1, u_2, u_3 \rangle$

$\vec{v} = \langle v_1, v_2, v_3 \rangle$

$\vec{a} = \langle a_1, a_2, a_3 \rangle$

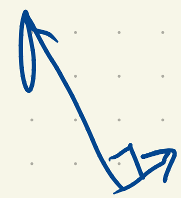
$\|\vec{a}\|^2 = a_1 \cdot a_1 + a_2 a_2 + a_3 a_3$

$\vec{a} \cdot \vec{a} = \vec{a}^2$

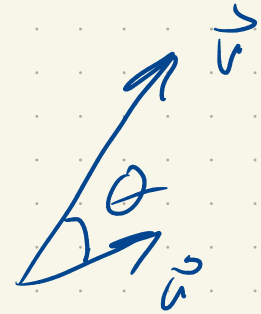
$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$

$\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$

$\vec{u} \cdot \vec{v} = 0 \qquad \vec{u} \perp \vec{v}$



$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$



$0 \leq \theta \leq \pi$

$0^\circ \leq \theta \leq 180^\circ$

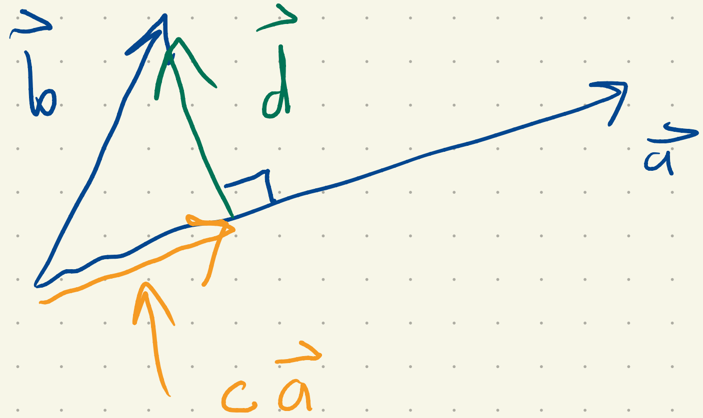
$\vec{u} \cdot \vec{v} > 0 \Rightarrow$ acute angle

$\vec{u} \cdot \vec{v} < 0 \Rightarrow$ angle obtuse

work done by force $\vec{F}$ over a displacement $\overrightarrow{PQ}$

is $\vec{F} \cdot \overrightarrow{PQ}$.

Orthogonal Projection



$$\vec{b} = c\vec{a} + \vec{d}$$

$$\vec{b} \cdot \vec{a} = c\, \vec{a} \cdot \vec{a} + \underbrace{\vec{d} \cdot \vec{a}}$$

$$\vec{b} \cdot \vec{a} = c\, \|\vec{a}\|^2 + 0 \Rightarrow c = \frac{\vec{b} \cdot \vec{a}}{\|\vec{a}\|^2}$$

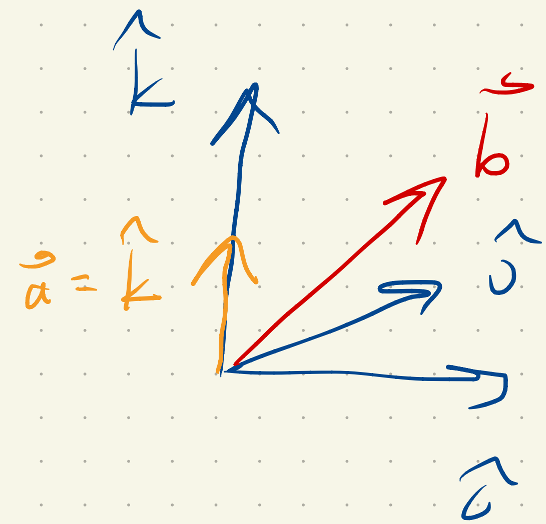
$$\boxed{\frac{\vec{b}\cdot\vec{a}}{\|\vec{a}\|^2}\,\vec{a}} \rightarrow \text{orthogonal projection of } \vec{b} \text{ onto } \vec{a}$$

$$\rightarrow \operatorname{proj}_{\vec{a}} \vec{b}$$

$$\vec{b} = 5\hat{\imath} + 2\hat{\jmath} - 6\hat{k}$$

$$\vec{a} = \vec{k}$$

$$\operatorname{proj}_{\vec{a}} \vec{b} \overset{?}{=} -6\hat{k}$$



$$\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|^2}\vec{a}$$

$$\vec{a}\cdot\vec{b} = (5\hat{i}+2\hat{j}-6\hat{k})\cdot\hat{k}$$

$$= 5\hat{i}\cdot\hat{k}+2\hat{j}\cdot\hat{k}-6\hat{k}\cdot\hat{k}$$

$$= 5\cdot 0 + 2 0 - 6\|\hat{k}\|^2$$

$$= -6$$

$$\hat{k} = \langle 0, 0, 1\rangle$$

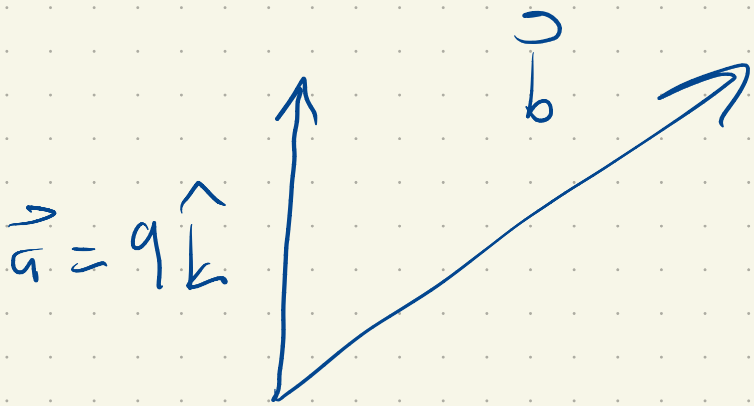
$$\|\hat{k}\|^2 = 0^2+0^2+1^2 = 1$$

$$\|\vec{a}\|^2 = 1$$

$$= \frac{-6}{1}\hat{k} = -6k$$

$$\vec{b} = 5\hat{i} + 2\hat{j} - 6\hat{k}$$

$$\vec{a} = 9\hat{k}$$



$$\text{proj}_{\vec{a}}\,\vec{b}$$

$$\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|^2}\,\vec{a} = \frac{9\cdot(-6)}{81}\,9\hat{k} = \frac{81\cdot(-6)}{81}\,\hat{k} = -6k$$

Section 2.4 Cross Product

2x2 determinant (Warmup exercise)

$\vec{u} = \langle u_1, u_2 \rangle$

$\vec{v} = \langle v_1, v_2 \rangle$

2x2 matrix

$$\begin{bmatrix} \vec{u} \\ \vec{v} \end{bmatrix} := \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}$$

2x2 determinant

By definition

$$\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = u_1 v_2 - u_2 v_1$$

1) $\begin{bmatrix} \vec{u} \\ \vec{u} \end{bmatrix}$ $\qquad$ $\begin{vmatrix} \vec{u} \\ \vec{u} \end{vmatrix} = \begin{vmatrix} u_1 & u_2 \\ u_1 & u_2 \end{vmatrix} = u_1 u_2 - u_2 u_1$

$= 0$

2) $\begin{vmatrix} \vec{u} \\ \vec{v} \end{vmatrix}$ vs $\begin{vmatrix} \vec{v} \\ \vec{u} \end{vmatrix}$

$$\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = u_1 v_2 - u_2 v_1, \quad \begin{vmatrix} v_1 & v_2 \\ u_1 & u_2 \end{vmatrix} = v_1 u_2 - v_2 u_1$$

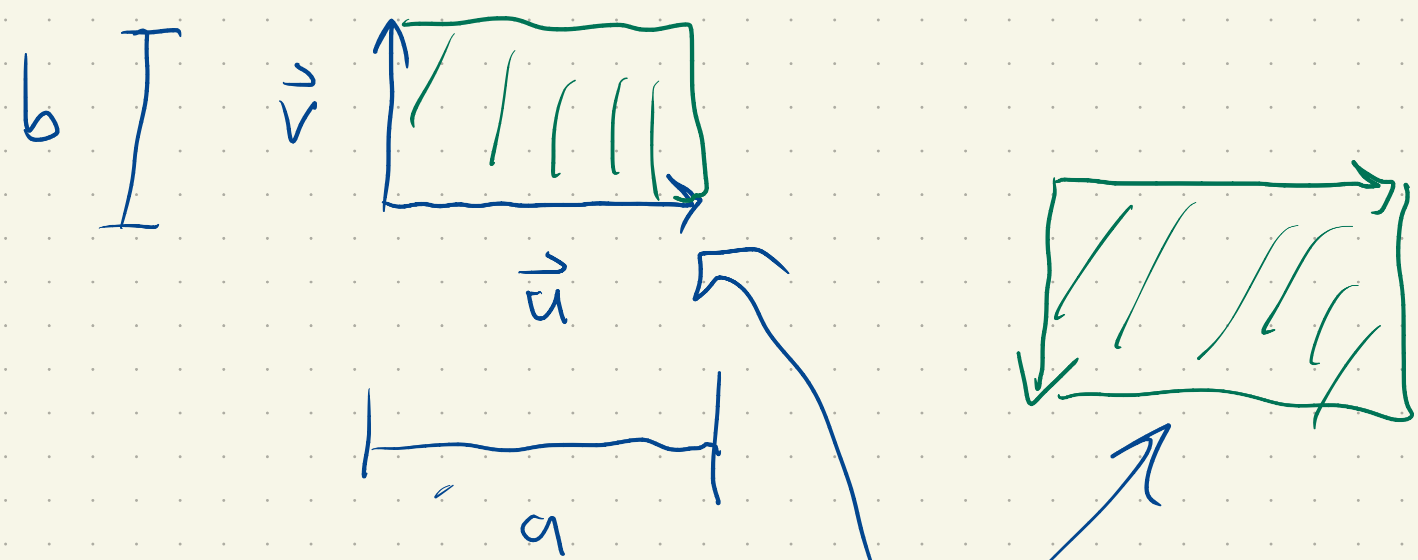
$$\begin{vmatrix} \vec{u} \\ \vec{v} \end{vmatrix} = - \begin{vmatrix} \vec{v} \\ \vec{u} \end{vmatrix}$$

3) $\vec{u} = \langle a, 0 \rangle$

$\vec{v} = \langle 0, b \rangle$

$$\begin{vmatrix} \vec{u} \\ \vec{v} \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab - 00$$

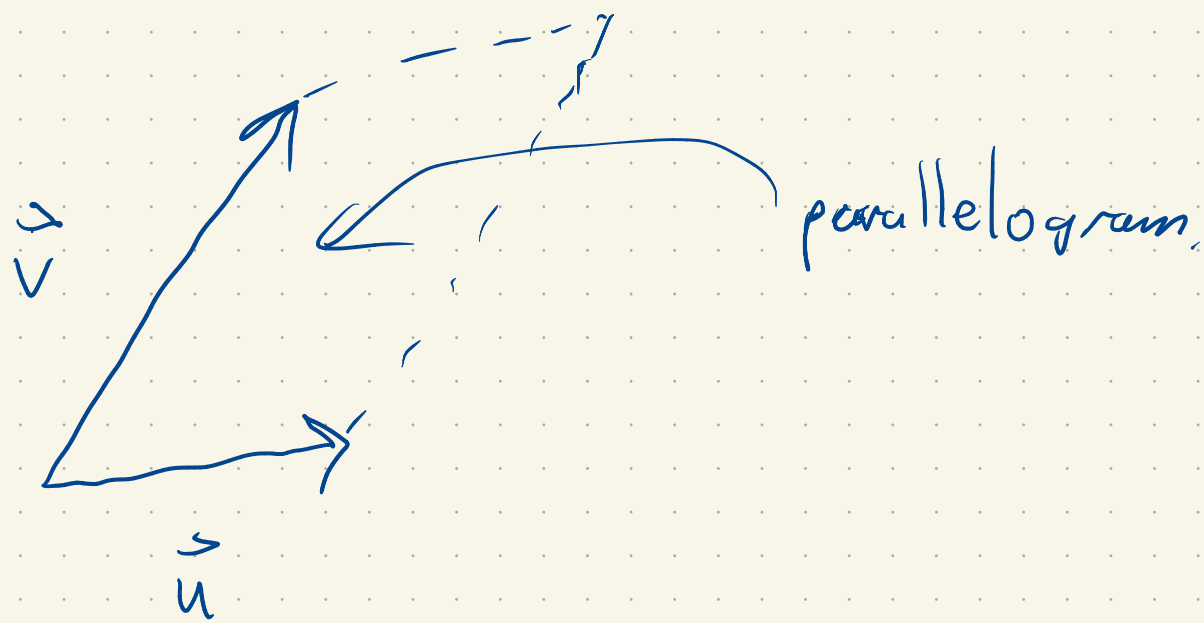
$$= ab$$



$|ab|$ is the area of this rectangle

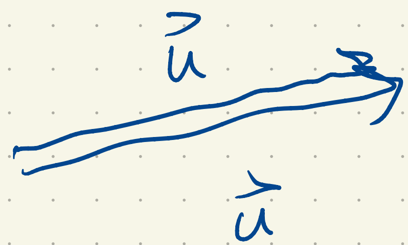
Fact: For all 2-d vectors

$\begin{vmatrix} \vec{u} \\ \vec{v} \end{vmatrix}$ is, up to sign, the area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$.



The result is positive if you turn left to go from $\vec{u}$ to $\vec{v}$. It's negative if you turn right.

$$\begin{vmatrix} \vec{u} \\ \vec{u} \end{vmatrix} = 0$$



3-d version

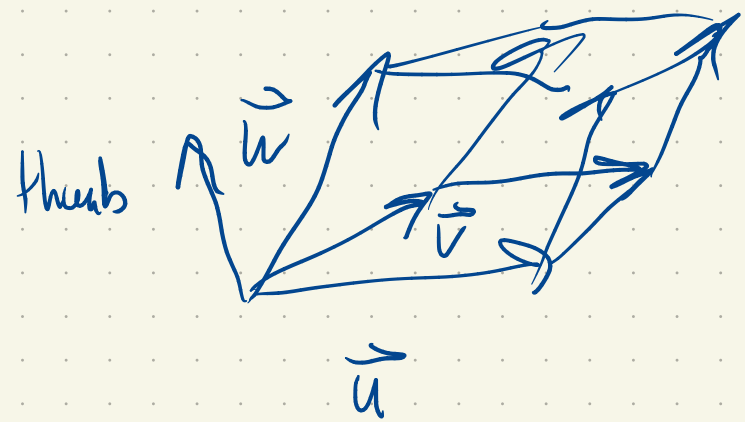
$$\vec{u} = \langle u_1, u_2, u_3 \rangle$$

$$\vec{v} = \langle v_1, v_2, v_3 \rangle$$

$$\vec{w} = \langle w_1, w_2, w_3 \rangle$$

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

$\begin{vmatrix} \vec{u} \\ \vec{v} \\ \vec{w} \end{vmatrix}$ is up to sign the volume of the parallelapiped spanned by $\vec{u}, \vec{v}, \vec{w}$



sign is detemined by which side of plane spaned by $\vec{u}, \vec{v}$ that $\vec{w}$ lies on.

Cross Product

$$\vec{a} = \langle a_1, a_2, a_3 \rangle$$

$$\vec{b} = \langle b_1, b_2, b_3 \rangle$$

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$$

$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}$$

Why??

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$$

$$\vec{a} \cdot (\vec{a} \times \vec{b}) =$$

$$a_1 \cdot (\cancel{a_2 b_3} - \cancel{a_3 b_2}) + a_2 (\cancel{a_3 b_1} - \cancel{a_1 b_3}) + a_3 (\cancel{a_1 b_2} - \cancel{a_2 b_1})$$

$$= 0$$