$\vec{v} = \langle \sqrt{5}, -2, 4 \rangle$

unit vector

Task: find a vector $\vec{u}$ of unit length pointing in the same direction as $\vec{v}$

$$\|\vec{v}\|^2 = (\sqrt{5})^2 + (-2)^2 + 4^2$$
$$= 5 + 4 + 16$$
$$= 25$$

$$\|\vec{v}\| = 5$$

$$\vec{u} = \frac{1}{5}\langle \sqrt{5}, -2, 4 \rangle$$
$$= \left\langle \frac{\sqrt{5}}{5}, -\frac{2}{5}, \frac{4}{5} \right\rangle$$

Divide $\vec{v}$ by its length. $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|}$

Common unit vectors

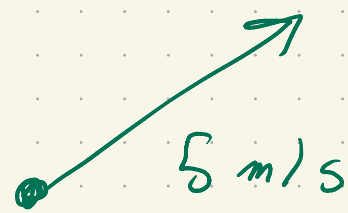
$$\hat{i} = \langle 1, 0, 0 \rangle \qquad \left(\hat{i} = \langle 1, 0 \rangle \right.$$
$$\hat{j} = \langle 0, 1, 0 \rangle \qquad \left. \hat{j} = \langle 0, 1 \rangle \right)$$
$$\hat{k} = \langle 0, 0, 1 \rangle$$

standard basis vectors $\vec{0}$

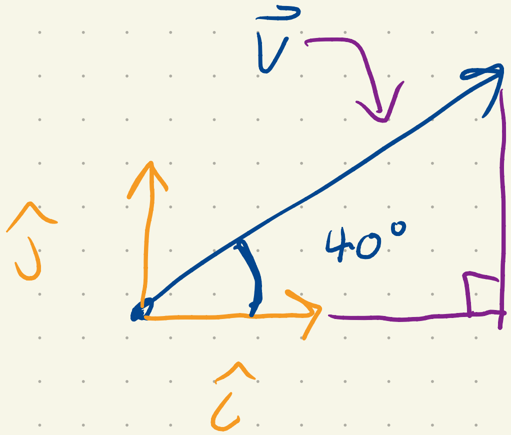
$$\vec{a} = \langle a_1, a_2, a_3 \rangle$$
$$= a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$$

Other vectorial quantities

- velocity m/s



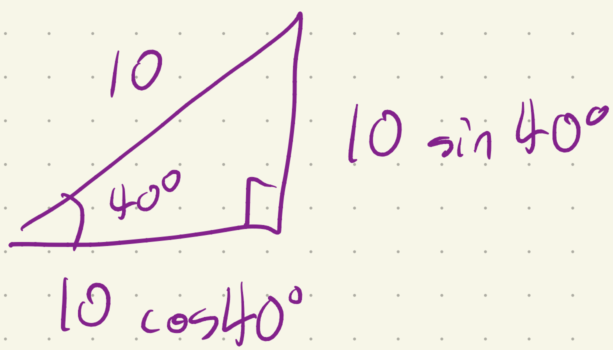
- acceleration m/s^2
- force $kg\, m/s^2$ (N)



$$|\vec{v}| = 10 \text{ km/h}$$

$$\vec{v} = v_1 \hat{i} + v_2 \hat{j}$$

$$= \langle v_1, v_2 \rangle$$



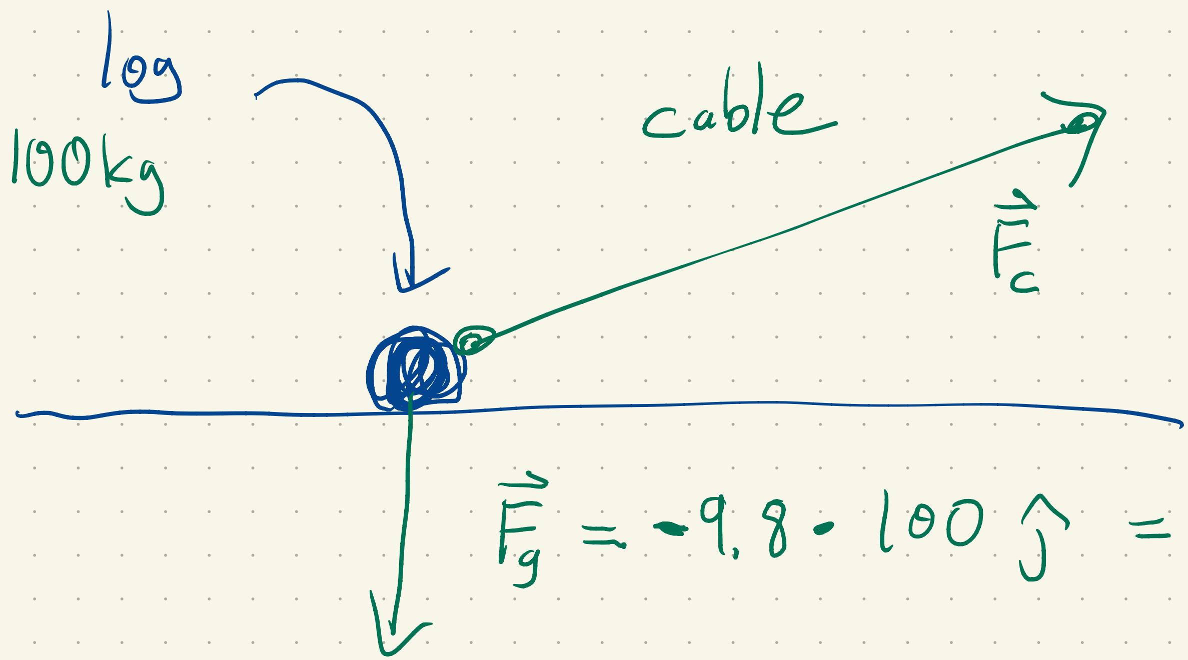
$$v_1 = 10 \cos(40°) \approx 7.7$$

$$v_2 = 10 \sin(40°) \approx 6.4$$

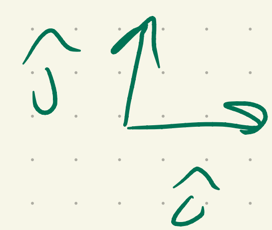
$$\vec{v} \approx 7.7\hat{i} + 6.4\hat{j}$$

When multiple forces act on an object the net force is the vector sum of the forces.

$$\vec{F}_c = 700\hat{\imath} + 500\hat{\jmath}$$

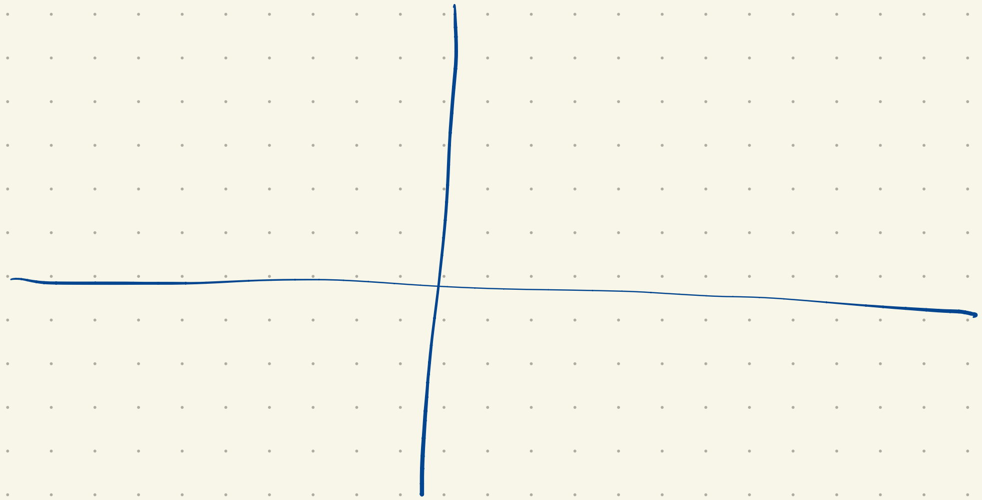


$$\vec{F}_g = -9.8 \cdot 100\,\hat{\jmath} = -980\,\hat{\jmath}\ N$$



$$F = \vec{F}_c + \vec{F}_g$$

$$= 700\hat{\imath} + 500\hat{\jmath} - 980\hat{\jmath} = 700\hat{\imath} - 480\hat{\jmath}\ N$$



2.3 Dot Product

Def: $\vec{a} = \langle a_1, a_2, a_3 \rangle$

$\vec{b} = \langle b_1, b_2, b_3 \rangle$

$$\vec{a} \cdot \vec{b} = \underbrace{a_1 b_1 + a_2 b_2 + a_3 b_3}_{\text{number (scalar)}}$$

$$\vec{a} \cdot \vec{a} = a_1 \cdot a_1 + a_2 \cdot a_2 + a_3 \cdot a_3$$

$$= a_1^2 + a_2^2 + a_3^2$$

$$= \|\vec{a}\|^2$$

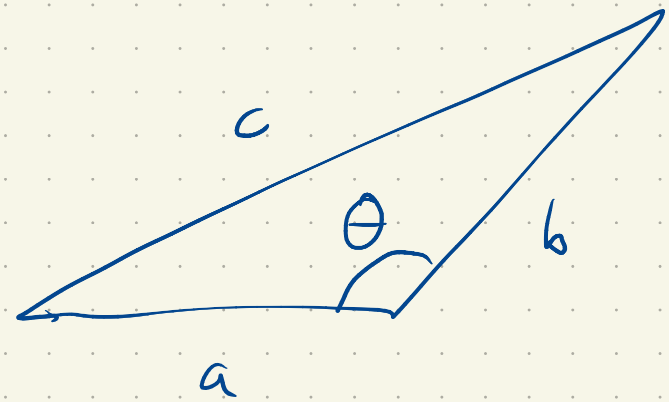
$$\|\vec{a} - \vec{b}\|^2 = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b})$$

$$= \vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b}$$

$$= \vec{a} \cdot \vec{a} - 2\, \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b}$$

$$= \|\vec{a}\|^2 - 2\, \vec{a} \cdot \vec{b} + \|\vec{b}\|^2$$

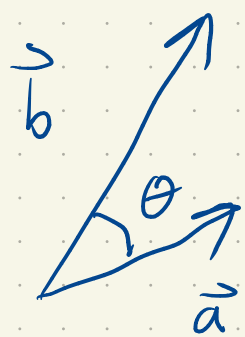
$$2\, \vec{a} \cdot \vec{b} = \|\vec{a}\|^2 + \|\vec{b}\|^2 - \|\vec{a} - \vec{b}\|^2$$

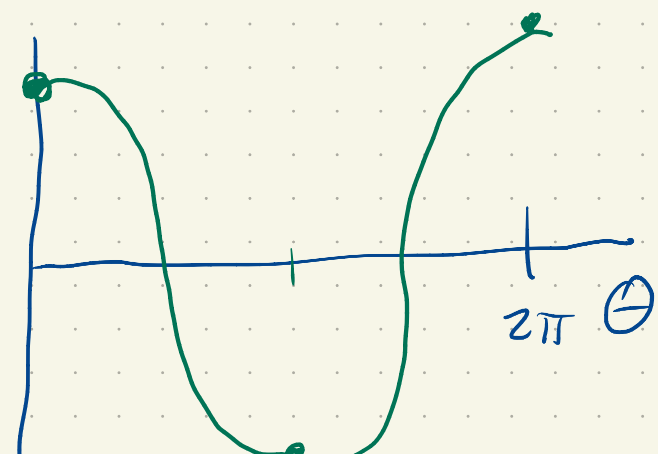


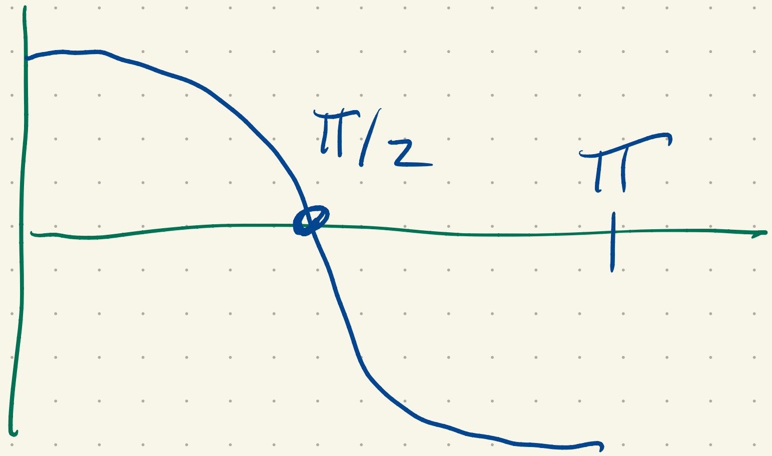
$$2ab\cos\theta = a^2 + b^2 - c^2$$

$$\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta$$

$0 \leq \theta \leq \pi$







$\langle 1, 7, 9 \rangle$

$\langle -2, 1, 3 \rangle$

$\cos\theta$

$\hat{i} \cdot \hat{j} = 0$

$\|\hat{i}\| \|\hat{j}\| \cos\theta$

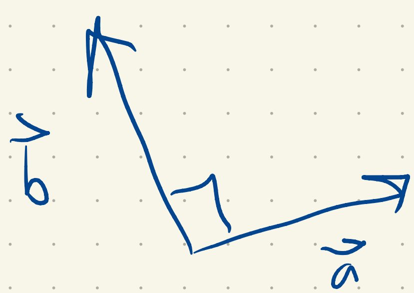
$\hat{i} = \langle 1, 0, 0 \rangle$

$\hat{j} = \langle 0, 1, 0 \rangle$

$\vec{a} \cdot \vec{b} = 0$ if and only if $\vec{a}$ and $\vec{b}$ are perpendicular

$\hat{j} \cdot \hat{k} = 0$

$\hat{j} \cdot \hat{j} = 1^2 = \|\hat{j}\|^2$



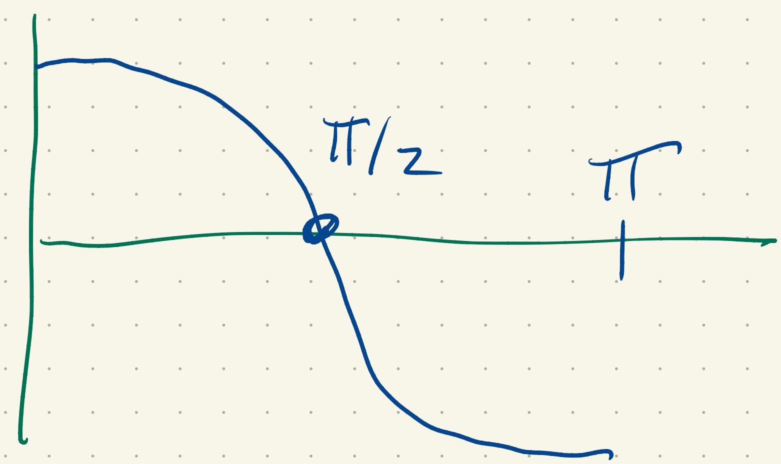
Fundamental Property of Dot Product

$$\vec{a} \cdot \vec{b} = 0 \quad \text{iff} \quad \vec{a} \perp \vec{b}$$

Secondary Property

$$\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$$

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \cos\theta$$



$\vec{a} \cdot \vec{b} > 0$

$\Downarrow$

angle between $\vec{a}$ and $\vec{b}$ is acute

$\|\vec{a}\| \|\vec{b}\| \cos\theta > 0$

$\Rightarrow \cos\theta > 0$

$\vec{a} \cdot \vec{b} < 0 \Rightarrow$ angle is obtuse

$\vec{a} \cdot \vec{b} = 0 \Rightarrow$ right angle

$\vec{a} = \langle 1, 2, 3 \rangle$

$\vec{b} = \langle -1, 2, 1 \rangle$

Task: compute the angle between $\vec{a}$ and $\vec{b}$.

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \cos\theta$$

$$\|\vec{a}\|^2 = 1^2 + 2^2 + 3^2 = 14$$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \, \|\vec{b}\|}$$

$$\|\vec{b}\|^2 = 6$$

$$\vec{a} \cdot \vec{b} = 1 \cdot (-1) + 2 \cdot 2 + 3 \cdot 1$$

$$= -1 + 4 + 3$$

$$= 6$$

$$\cos\theta = \frac{6}{\sqrt{14} \cdot \sqrt{6}} = \sqrt{\frac{6}{14}}$$

$$\theta = \arccos\left(\sqrt{6/14}\right) = 49.1^\circ$$

Dot Products and Physics

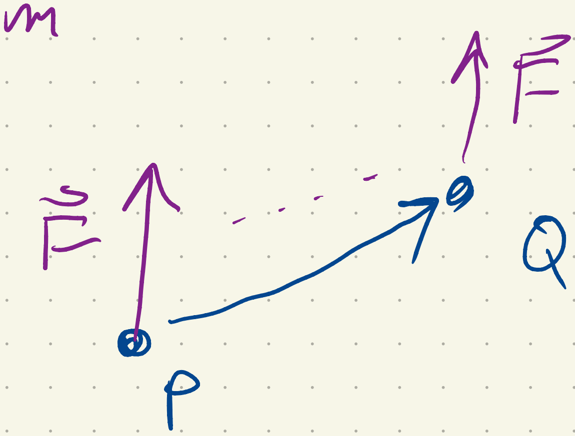
If a constant force $\vec{F}$ is applied to a body that moves from P to Q then the body gains / loses energy

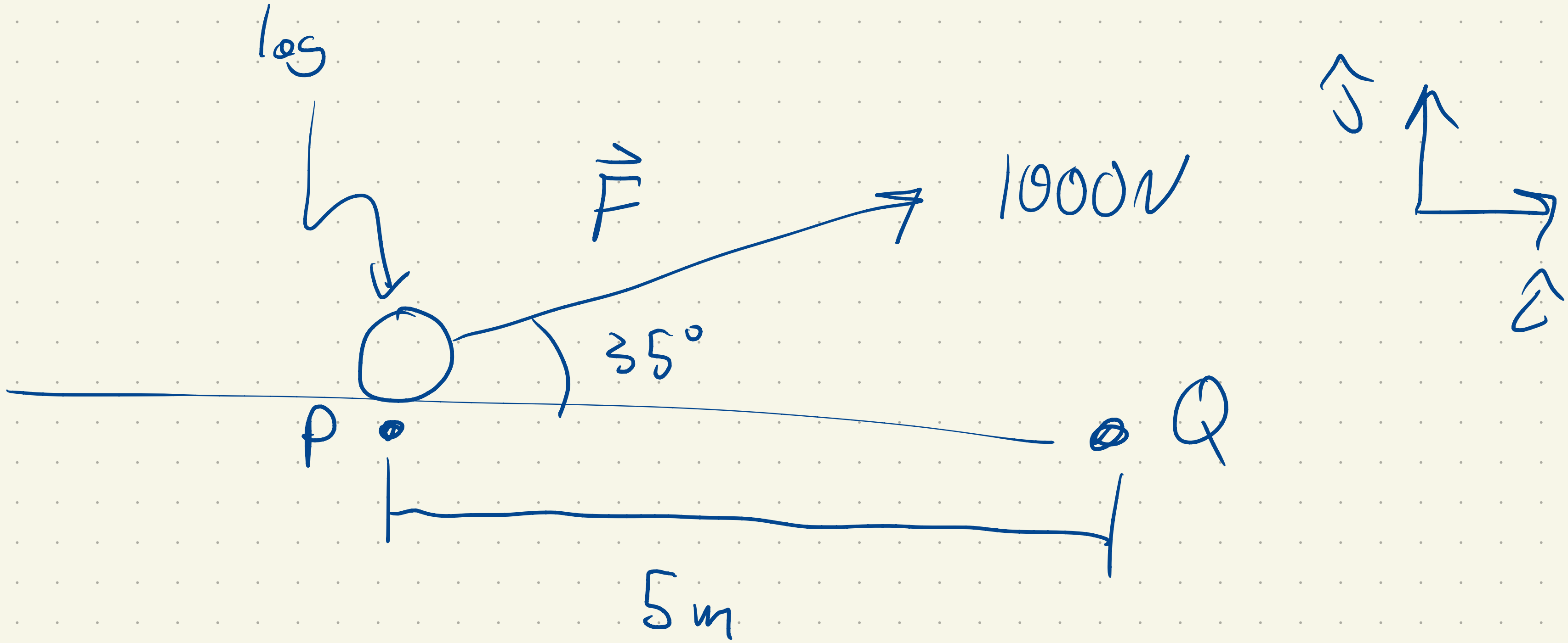
This change is the <u>work</u> done on the body.

Work is a change in energy, so is a scalar.

$$\text{work} = \vec{F} \cdot \overrightarrow{PQ}$$

- work: energy, joules, $kg\, m^2/s^2$
- $\vec{F}$: N, $kg\, \frac{m}{s^2}$
- $\overrightarrow{PQ}$: m





Task: compute the work done.

$$\overrightarrow{PQ} = 5\,\hat{\imath}$$

$$\vec{F} = 1000\cos 35^\circ\,\hat{\imath} + 1000\sin 35^\circ\,\hat{\jmath}$$

Work: $\vec{F}\cdot\overrightarrow{PQ} = 5000\cos 35^\circ$

~~= 819 J~~

$\hat{\imath}\cdot\hat{\imath} =$

$= 4095 \text{ J}$