$$\chi_G f_n \to \chi_G f$$

$$\int \chi_G f_n \to \int \chi_G f$$

$$\int f_n \to \int f$$

Goal: L^1 is complete

Every $\underbrace{\text{absolutely convergent series}}$ is convergent.

$$\sum_{n=1}^{\infty} [f_n] \qquad \sum_{n=1}^{\infty} \| [f_n] \| < \infty$$

$$\sum_{n=1}^{\infty} \int |f_n| \quad \text{is finite.}$$

$$g = \sum_{n=1}^{\infty} |f_n|$$

$$s_m = \sum_{n=1}^{m} |f_n|$$

$$\int |g| < \infty$$

Claim: $g \in L^1$ proof:

$$\int g < \infty$$

$$s_m \geq 0 \qquad s_m \nearrow g$$

$$\int g = \lim_{m \to \infty} \int s_m = \lim_{m\to\infty} \int \sum_{n=1}^{m} |f_n|$$

$$= \lim_{m\to\infty} \sum_{n=1}^{m} \int |f_n|$$

MCT

$$= \sum_{n=1}^{\infty} \|[f_n]\|_1 < \infty.$$

$$\lim_{n\to\infty} \int s_n = \int g$$

$$\int s_n = \int \sum_{k=1}^{n} |f_k| = \sum_{k=1}^{n} \int |f_k| = \sum_{k=1}^{n} \|[f_k]\|_1$$

$$\lim_{n\to\infty} \int s_n = \sum_{k=1}^{\infty} \|[f_k]\|_1 < \infty.$$

Let $r_n = \sum_{k=1}^{n} f_k$.

Observe $|r_n| \leq \sum_{k=1}^{n} |f_k| \leq g$

If for some x, $g(x) < \infty$ then

$$\sum_{k=1}^{\infty} f_k(x)$$ is absolutely convergent and converges to a limit. $\left(\sum_{k=1}^{\infty} |f_k(x)| = g(x) \right)$

We define $f = \begin{cases} \sum_{k=1}^{\infty} f_k(x) & g(x) < \infty \\ 0 & g(x) = \infty \end{cases}$

Exercise: f is measurable.

Observe that $|f| \leq g$ and $r_n \to f$ p.w. a.e.

By the Dominated Convergence Theorem $f \in L^1_{prev}$ and $\int r_n \to \int f$.

Observe $\| [r_n] - [f] \|_1 = \int |r_n - f|$.

Since $|r_n - f| \leq 2g$ and since $|r_n - f| \to 0$ pw a.e.

then $\int |r_n - f| \longrightarrow \int 0 = 0$.

That is $\| [r_n] - [f] \|_1 \longrightarrow 0$ and $[r_n] \to [f]$.

Interesting dense subsets.

Thm Let $f \in L^1$. Given $\varepsilon > 0$ there exists:

a) an integrable simple function ψ such that

$$\int |f - \psi| < \varepsilon. \qquad \left(\|f - \psi\|_1 < \varepsilon \right),$$

b) a continuous function g with compact support ($g = 0$ outside a bounded set)

with $\int |f - g| < \varepsilon$.

Pf: Let $\varepsilon > 0$. Let $I_n = [-n, n]$. Suppose $f \in L^1$.

By the monotone convergence theorem $\int_{I_n^c} |f| \to 0$ and hence there exists an interval $I = I_n$ such that $\int_{I_n^c} |f| < \frac{\varepsilon}{4}$.

By the basic construction there exists a sequence of simple functions φ_n with $0 \le |\varphi_n| \le \chi_I |f|$ and $\varphi_n \to \chi_I f$ pointwise.

Moreover, $|\chi_I f - \varphi_n| \le 2\chi_I |f|$ and $\chi_I f - \varphi_n \to 0$ pointwise a.e. By the DCT $\int |\chi_I f - \varphi_n| \longrightarrow 0$.

So we can find a simple function φ with $0 \le |\varphi| \le \chi_I |f|$ and $\int |\chi_I f - \varphi| < \varepsilon/4$.

Note that $\int |f - \varphi| = \int_{I^c} |f| + \int_I |f - \varphi| = \int_{I^c} |f| + \int |\chi_I f - \varphi|$

$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon.$$

Since φ is integrable, we have proved part a.

Pick K such that $|\varphi| \leq K$.

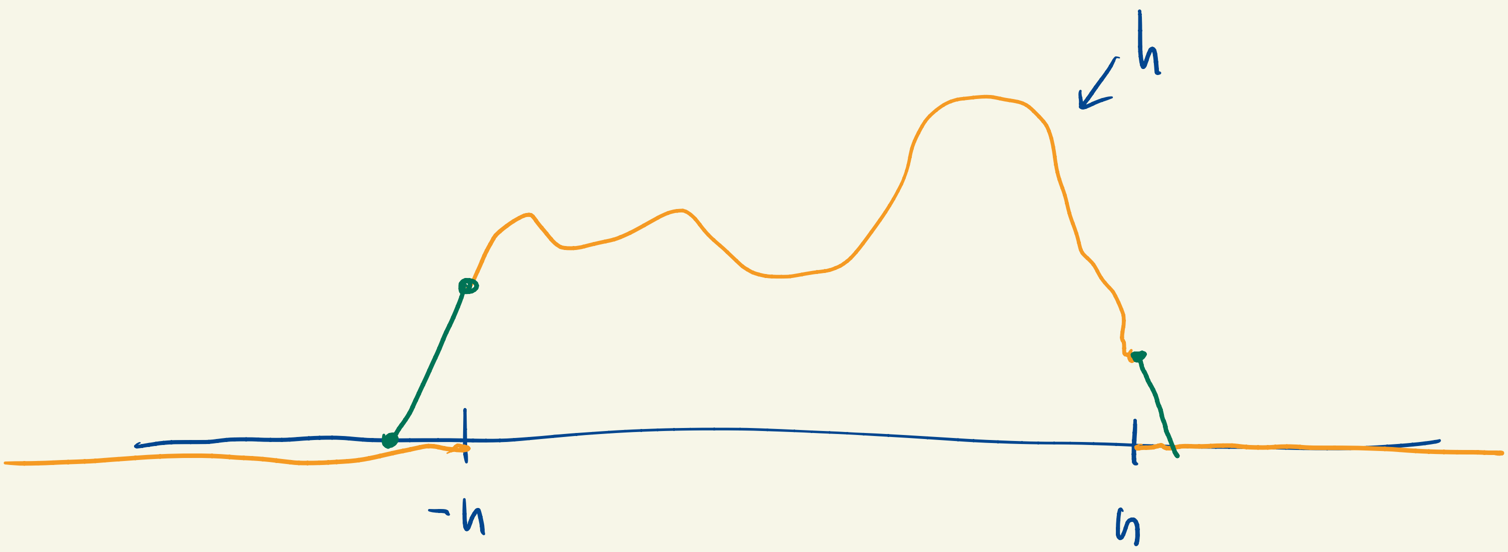
By your homework there is a continuous function h on I ($|h| \leq K$) such that $m(\{\varphi \neq h\}) < \varepsilon/8K$.

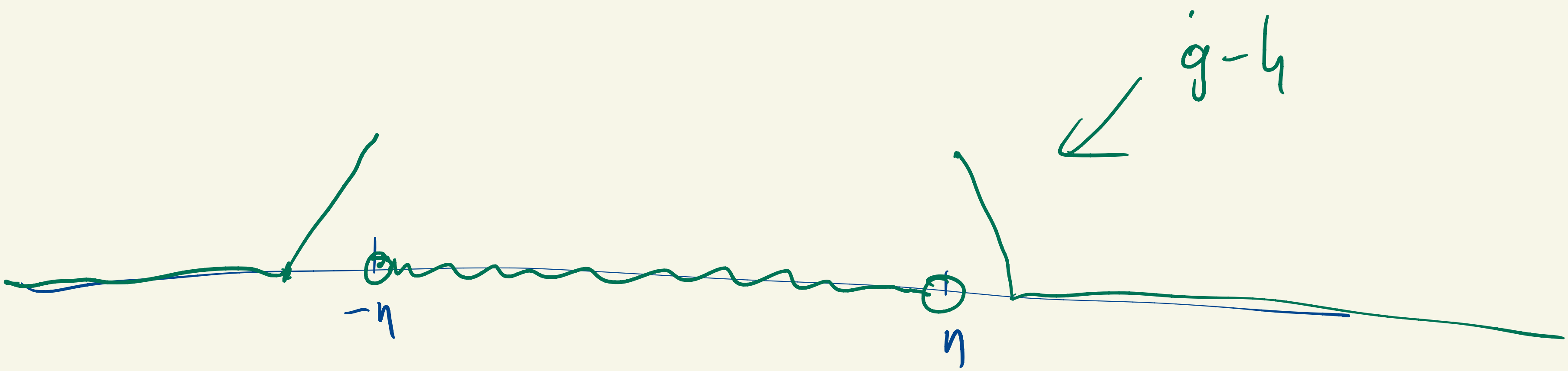
Observe $\int_I \underbrace{|\varphi - h|}_{\leq 2K \chi_{\{\varphi \neq h\}}} \leq 2K\, m(\{\varphi \neq h\}) \leq \frac{\varepsilon}{4}$.

We extend h by 0 outside of I. Observe

$$\int |f - h| \leq \int |f - \varphi| + \int |\varphi - h| < \frac{\varepsilon}{2} + \int_I |\varphi - h| < \frac{3\varepsilon}{4}.$$

The proof is done noting that we can find a continuous function g that vanishes outside of an interval such that $\int |h-g| < \frac{\varepsilon}{4}$.





Exercise: Polynomials are dense in $L^1([a,b])$

Exercise: $L^1([a,b])$ is separable.

Exercise: L^1 is separable. $L^1([-n,n])$

Exercise: Piecewise linear compactly supported functions are dense in L^1

L^p spaces $(1 \le p \le \infty)$

$$L^p = \{ f : \text{measurable}, |f|^p \in L^1 \}$$

(in truth, these are equivalence classes)

$$\|f\|_p = \left[\int |f|^p \right]^{1/p}$$

$\to$ $p < \infty$

a) L^p is a vector space

b) the space is complete

c) $\boxed{\text{If } f \in L^p,\ g \in L^q \quad \frac{1}{p} + \frac{1}{q} = 1 \quad \text{then} \quad fg \in L^1 \quad \int |fg| \leq \|f\|_p \|g\|_q}$

Hölder's Ineq. ($p = 2$, $q = 2$ also and this is Cauchy Schwartz)

d) continuous functions with compact support are dense in L^p ($p \neq \infty$).

that vanish outside some interval

$L^\infty = \{ f : \text{measurable and there exists } K \text{ with } |f| \leq K \text{ a.e.} \}$

$\|f\|_\infty = \inf \{ K : |f| \leq K \text{ a.e.} \}$

"$\max |f|$"