Recall: If f is meas and $f \geq 0$ and

$$\int f = 0 \implies f = 0 \text{ a.e.}$$

Exercise: If $f \geq 0$ and $f = 0$ a.e. then $\int f = 0$

(Do this from scratch)

Exercise: If f, g are measurable and g is finite everywhere then $f + g$ is measurable.

Lemma: Suppose $f, g \geq 0$ are measurable and integrable. Then

$f = g$ a.e. iff $\int_E f = \int_E g$ for all measurable sets E.

Pf: Suppose $f = g$ a.e. Let $N = \{f \neq g\}$. If E is measurable then

$$\int_E f = \int_{E\cap N} f + \int_{E \cap N^c} f = \int_{E\cap N^c} f = \int_{E \cap N^c} g = \int_E g.$$

$$\int \chi_{E\cap N} f \qquad \int_E \chi_N f \qquad \int_E \chi_{E^c} f$$

For the converse consider the set $E = \{f > g\}$. (Exercise: this is measurable)

On E, $f = (f-g) + g$. Hence

finite.

$$\int_E f = \int_E \underbrace{(f-g)}_{\text{meas.}} + \int_E g$$

and hence $\int_E (f-g) = 0$.

Now $\chi_E(f-g) \geq 0$ and $\int \chi_E(f-g) = 0$ so

$\chi_E(f-g) = 0$ a.e. and hence $m(E) = 0$.

Similarly, $m(\{g > f\}) = 0$ and $f = g$ a.e.

$f \geq 0$

Prop: If $f, g \in L^1$ then

$f = g$ a.e. iff $\int_E f = \int_E g$ for all measurable sets E.

Pf: $\boxed{\text{Exercise: If } f=g \text{ a.e. then } \int_E f = \int_E g \text{ for all meas. } E.}$

Suppose $\int_E f = \int_E g$ for all meas. sets E.

Let $E_{++} = \{ f \geq 0, g \geq 0 \}$. Then for any measurable set F

$$\int_F \chi_{E_{++}} f = \int_{F \cap E_{++}} f = \int_{F \cap E_{++}} g = \int_F \chi_{E_{++}} g .$$

Thus, by the previous lemma, $\chi_{E_{++}} f = \chi_{E_{++}} g$ a.e.

and $f = g$ a.e. on E_{++}.

Similarly $f = g$ a.e. on $E_{--} = \{ f < 0, g < 0 \}$.

Consider $E_{+-} = \{ f \geq 0, g < 0 \}$. Then

$$0 \leq \int_{E_{+-}} f = \int_{E_{+-}} g \leq 0 .$$

Hence $\int_{E_{+-}} g = 0$ and $\chi_{E_{+-}} g = 0$ a.e.

Hence E_{+-} is a null set. Similarly $E_{-+} = \{f<0, g\geq 0\}$ is null. But then $\{f \neq g\}$ is a null set; it is the union of E_{-+}, E_{+-} and two null subsets of E_{++} and E_{--}.

Change of notation $L^1(E) \longmapsto L^1_{prov}(E)$

Def: $L^1(E)$ where E is measurable consists of equivalence classes of functions in $L^1_{prov}(E)$ where

$f \sim g$ if $f = g$ a.e.

Exercise: This is an equivalence relation.

Exercise: If $f \in L^1_{prov}(E)$ and $g = f$ a.e. then $g \in L^1_{prov}(E)$.

For now we'll write $[f]$ for elements of L^1 where $f \in L^1_{prov}$.

If $[f] \in L^1(\mathbb{R})$ what is

$$\int_E [f] = \int_E f$$

If $\hat{f} = f$ a.e. $\quad [\hat{f}] = [f]$

$$\int_E \hat{f} \stackrel{?}{=} \int_E f$$

$[f+g]$

How to add? $\quad [f] + [g] = [\hat{f} + \hat{g}]$

where $\hat{f} = f$ a.e. and is finite everywhere and similarly for $\hat{g}$.

Exercise: This is well defined.

$$\int([f]+[g]) = \int [\hat{f}+\hat{g}]$$

$$= \int (\hat{f}+\hat{g})$$

$$= \int \hat{f} + \int \hat{g}$$

$$= \int [\hat{f}] + \int [\hat{g}]$$

$$= \int [f] + \int [g]$$

$$\underset{\in\mathbb{R}}{c}\,[f] := [cf].$$ Exercise: this is well defined.

Exercise: $\int c[f] = c\int [f]$

Exercise: L^1 is a vector space under these operations, and

$$[f] \longmapsto \int_E [f] \quad \text{is linear in } E.$$

Def: $\big|[f]\big| = [|f|]$

Exercise: this is well defined.

Def: If $[f] \in L^1$

$$\|[f]\|_1 = \int \big|[f]\big| = \int |f|$$

Is this a norm?

$$\|[f]+[g]\|_1 = \|[\hat{f}+\hat{g}]\|_1$$

$$= \int |\hat{f}+\hat{g}|$$

$$\leq \int (|\hat{f}|+|\hat{g}|)$$

$$= \int |\hat{f}| + \int |\hat{g}|$$

$$= \|[\hat{f}]\|_1 + \|[\hat{g}]\|_1$$

$$= \|[f]\|_1 + \|[g]\|_1 \quad \checkmark$$

Suppose $\|[f]\|_1 = 0$.

Then $\int |f| = 0$.

Since $|f| \geq 0 \Rightarrow |f| = 0$ a.e. $\Rightarrow f = 0$ a.e.

$[f] = [0]$.

Next big step: L^1 is complete.

DCT: $\left.\begin{array}{l} g \in L^1_{prov} \\ |f_n| \leq g \\ f_n \to f \quad \text{p.w.} \end{array}\right] \Rightarrow \begin{array}{l} f \in L^1_{prov} \\ \int f_n \to \int f \end{array}$

$g \in L^1_{pw}$

Modification: $|f_n| \leq g$ a.e. for each n

$f_n \to f$ p.w. a.e.

$$\Rightarrow \quad \int f_n \to \int f$$

Let $E_n = \{ |f_n| > g \}$.

Let $F = \{ f_n \not\to f \}$

Let $G = ((\cup E_n) \cup F)^c$

$$|\chi_G f_n| \leq g$$

$$\chi_G f_n \to \chi_G f$$

$$\int \chi_G f_n \to \int \chi_G f$$

$$\int f_n \to \int f$$

Goal: L^1 is complete.

Every $\underbrace{\text{absolutely convergent series}}$ is convergent.

$$\sum_{n=1}^{\infty} [f_n] \qquad \sum_{n=1}^{\infty} \| [f_n] \| < \infty$$

$$\sum_{n=1}^{\infty} \int |f_n| \quad \text{is finite.}$$

$$g = \sum_{n=1}^{\infty} |f_n|$$

$$s_m = \sum_{n=1}^{m} |f_n|$$

Claim: $g \in L^1_{\text{prov}}$.

$$s_m \geq 0 \qquad s_m \nearrow g$$

$$\int g \underset{\text{MCT}}{=} \lim_{m \to \infty} \int s_m = \lim_{m \to \infty} \int \sum_{n=1}^{m} |f_n|$$

$$= \lim_{m \to \infty} \sum_{n=1}^{m} \int |f_n|$$

$$= \sum_{n=1}^{\infty} \| [f_n] \|_1 < \infty.$$