By the MCT $\lim_{n\to\infty} \int g_n = \int f$.

Note that for each n $g_n \le f_n$. Thus

$$\liminf_{n\to\infty} \int f_n \ge \liminf_{n\to\infty} \int g_n = \int f. \qquad \square$$

Observe: Fatou's Lemma implies the MCT.

$$\left.\begin{aligned} & f_n \uparrow f \qquad \int f_n \le \int f \\ & \int f \le \liminf_{n\to\infty} \int f_n = \lim_{n\to\infty} \int f_n \end{aligned}\right] \Rightarrow \int f = \lim_{n\to\infty} \int f_n$$

$f_n \geq 0$, meas

$$f = \sum_{n=1}^{\infty} f_n \qquad \int f = \sum_{n=1}^{\infty} \int f_n$$

seq of partial sums is monotone increasing + MCT.

Int of arbitrary meas functions.

$$f = f_+ - f_- \qquad f_+ = f \vee 0$$

$$f_- = (-f) \vee 0$$

$$\int f := \int f_+ - \int f_-$$

so long as at least one of the two integrals is finite.

If both are finite we $\boxed{\text{provisionally}}$ say $f \in L_1(\mathbb{R})$

and say f is integrable.

Note $\int |f| = \int (f_+ + f_-) = \int f_+ + \int f_-$

So f is integrable iff $\underbrace{\int |f|}$ is finite.

$\hookrightarrow$ not a norm on the integrable functions

$f = \chi_{\mathbb{Q}} \qquad \int |\chi_{\mathbb{Q}}| = 0$

Given $f, g \in L_1$ does $f+g$ even make sense?

Suppose $f, g \in L_1$ and are finite everywhere

Is $f+g \in L_1$? Yes!

② $f+g = (f+g)_+ - (f+g)_-$

$f+g = f_+ - f_- + g_+ - g_-$

$$\int \left[(f+g)_+ + f_- + g_-\right] = \int \left[(f+g)_- + f_+ + g_+\right]$$

$$\int (f+g)_+ + \int f_- + \int g_- = \int (f+g)_- + \int f_+ + \int g_+$$

$$\int (f+g)_+ - \int (f+g)_- = \int f_+ - \int f_- + \int g_+ - \int g_-$$

$$\int f+g = \int f + \int g$$

---

1 $$|f+g| \leq |f|+|g|$$

$$\int |f+g| \leq \int (|f|+|g|) = \int |f| + \int |g|$$

So if $f, g \in L_1$, $f+g \in L_1$ also.

- $\int cf$ if $f \in L^1$ and $c \in \mathbb{R}$

$$\int |cf| = \int |c||f| = |c| \int |f| \Rightarrow cf \in L^1$$

If $c \geq 0$

$$\int cf = \int (cf)_+ - \int (cf)_-$$

$$= \int c f_+ - \int c f_-$$

$$= c \int f_+ - c \int f_- = c \left[\int f_+ - \int f_- \right]$$

If $c = -1$

$$\int cf = \int (cf)_+ - \int (cf_-)$$

$$(-f)_+ = (-f) \vee 0$$

$$= \int (-f)_+ - \int (-f)_-$$

$$= \int f_- - \int f_+$$

$$= - \int f$$

Now combine.

Upshot: The finite everywhere elements of L_1 form a vector space and integration is linear on it.

How do the Riemann and Lebesgue integrals compare?

$$(R)\int_a^b f \qquad\qquad (L)\int_a^b f$$

In fact every Riemann integrable function is measurable and bounded and hence integrable and the two integrals coincide.

On current HW: you are showing this if $f \geq 0$.

$$\underbrace{f + A}_{} \geq 0 \quad \longrightarrow \quad \text{measurable}$$

$$(R)\int_a^b (f+A) = (L)\int_a^b (f+A)$$

$$\parallel$$

$$(R)\int_a^b f + \underbrace{(R)\int_a^b A}_{A(b-a)} = (L)\int_a^b f + \underbrace{(L)\int_a^b A}_{A(b-a)}$$

Our final convergence theorem.

We have seen $\int f_n \not\to \int f$

not always true

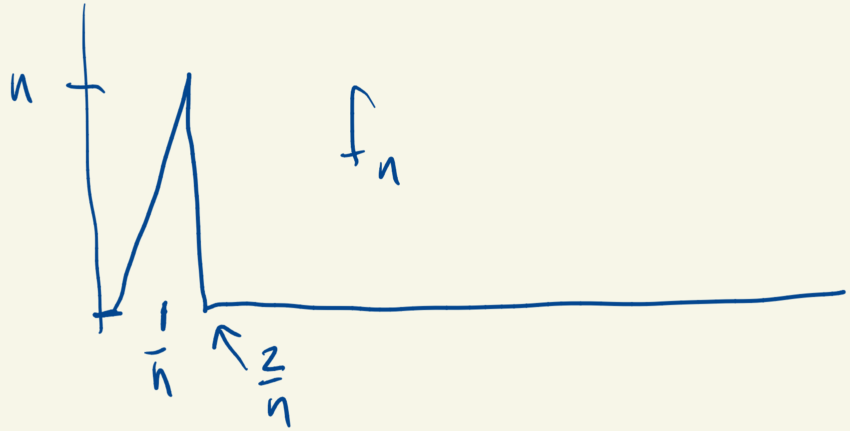
if $f_n \to f$ p.w.

$\frac{1}{n}$

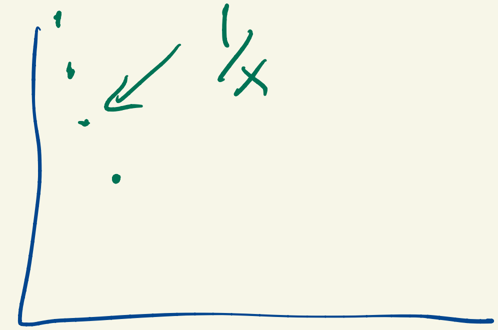
$\chi_{[1,\infty)} = f_n$

$\int f_n = \infty$

$\int f = 0$



$$\int f_n = 1$$



Dominated Convergence Theorem.

$\boxed{[\text{Everybody is finite everywhere}]}$

Suppose $g \geq 0$ is in $L_1(\mathbb{R})$.

Let (f_n) be a sequence of measurable functions such that $f_n \to f$ p.w.

for some f and such that $\underbrace{|f_n| \leq g}$ for each n.

Then $\lim_{n\to\infty} \int f_n = \int f$. $\longrightarrow$ $\int |f_n| \leq \int g$

Pf: Consider the functions $f_n + g \geq 0$. $(g \geq f_n \geq -g)$

$|f_n| \leq g$

$(|f| \leq g$ also$)$

Fatou's lemma implies

$$\liminf_{n\to\infty} \int (f_n+g) \geq \int f+g = \int f + \int g.$$

But $\liminf_{n\to\infty} \int (f_n+g) = \left(\liminf_{n\to\infty} \int f_n\right) + \int g$. Hence

$$\liminf_{n\to\infty} \int f_n \geq \int f.$$

$$\left(\begin{array}{l} -f_n \to -f \\ |-f_n| \leq g \end{array} \right)$$

By the same argument $\liminf_{n\to\infty} \int -f_n \geq \int -f.$

Hence $\limsup_{n\to\infty} \int f_n \leq \int f.$ We have therefore seen

$$\limsup_{n\to\infty} \int f_n \leq \int f \leq \liminf_{n\to\infty} \int f_n.$$

But $\liminf_{n\to\infty} \int f_n \leq \limsup_{n\to\infty} \int f_n$ so we have equality the limit exists and equals $\int f$.

$g(x) = \frac{1}{\sqrt{|x|}} \rightarrow$

Exercise: carefully show this function is integrable.

$$|f_n| \le g$$

$$\hat{f}_n \qquad \hat{g}$$

If $g \in L^1$ then

g is finite a.e.

$\int |g| \rightarrow$ finite.

$$\underbrace{\lim \int \hat{f}_n}_{\int f_n} = \int \hat{f}$$

$$\|$$

$$\int f$$