Pf: (of MCT)

Since $f_n \leq f$ for all n, $\int f_n \leq \int f$ for each n.

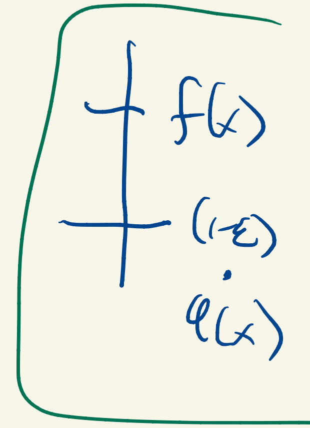
So $\lim\limits_{n\to\infty} \int f_n \leq \int f$.

It suffices then to show $\int f \leq \lim\limits_{n\to\infty} \int f_n$.

Recall $\int f = \sup \left\{ \int \varphi : \varphi \text{ is simple, integrable, } 0 \leq \varphi \leq f \right\}$.

Suppose φ is simple, integrable and $0 \leq \varphi \leq f$.

Let $1 > \varepsilon > 0$ and consider $(1-\varepsilon)\varphi$. Observe $(1-\varepsilon)\varphi < f$ everywhere $f \neq 0$. Let $E_n = \{ f_n \geq (1-\varepsilon)\varphi \}$ and observe that the E_n's are increasing and $\bigcup E_n = \mathbb{R}$



Now for each n

$$\int f_n \geq \int_{E_n} f_n \geq \int_{E_n} (1-\varepsilon)\varphi.$$

Thus

$$\lim_{n\to\infty} \int f_n \geq \lim_{n\to\infty} \int_{E_n} (1-\varepsilon)\varphi$$

$$= \int (1-\varepsilon)\varphi.$$

This is true for all $\varepsilon \in (0,1)$ and hence

$$\lim_{n\to\infty} \int f_n \geq \int \varphi. \text{ Hence } \lim_{n\to\infty} \int f_n \geq \int f.$$

Cor: Suppose the sets E_n are increasing and measurable and that $f \geq 0$ is measurable. Then

$$\lim_{n\to\infty} \int_{E_n} f = \int_E f \quad \text{where } E = \bigcup_n E_n.$$

Pf: Observe $\chi_{E_n} f$ is increasing to $\chi_E f$.

Now apply the MCT.

$$\int cf = c\int f \quad , \quad \int (f+g) = \int f + \int g$$

$c \geq 0.$ $\quad$ f, g are ≥ 0, meas.

Lemma: If f is non-negative and measurable there is a sequence of non-negative integrable simple functions φ_n with $0 \leq \varphi_n \leq f$ and $\varphi_n \uparrow f$.

Pf: By the Basic Construction there is a sequence of non negative simple functions ψ_n that increase to f.

Let $\varphi_n = \chi_{[-n,n]} \psi_n$. $\square$

Observe, in the language of the above

$$\int f = \lim \int \varphi_n \quad \text{by the MCT.}$$

Prop: If $f, g \geq 0$ and measurable, and if $c \geq 0$ then

$$\int cf = c \int f$$

$$\int (f+g) = \int f + \int g.$$

Pf: Let f_n, g_n be increasing non-negative simple functions converging

pointwise to f and g respectively.

Then: $f_n + g_n \to f+g$ pointwise and the sequence (f_n+g_n) is monotone increasing.

Thus, by the MCT

$$\int (f+g) = \lim_n \int (f_n+g_n) = \lim_n \left(\int f_n + \int g_n \right)$$

$$= \lim_{n\to\infty} \int f_n + \lim_{n\to\infty} \int g_n$$

$$= \int f + \int g .$$

The proof that $\int cf = c \int f$ is similar and easier.

What if (f_n) $f_n \geq 0$, meas.

$$f_n \downarrow f$$

$$\lim_{n\to\infty} \int f_n \overset{?}{=} \int f$$

Exercise: If $\int f_1$ is finite then indeed

$$\lim_{n\to\infty} \int f_n = \int f.$$

Exercise If $f_k \geq 0$, measurable and we set $F = \sum f_k$

then $\displaystyle \int F = \sum_{k=1}^{\infty} \int f_k$.

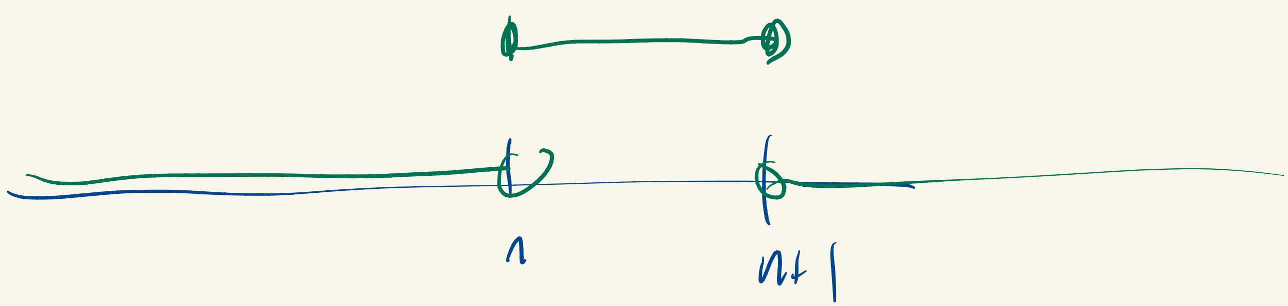
The MCT needs monotonicity. (and increasing).

What if you don't have it?

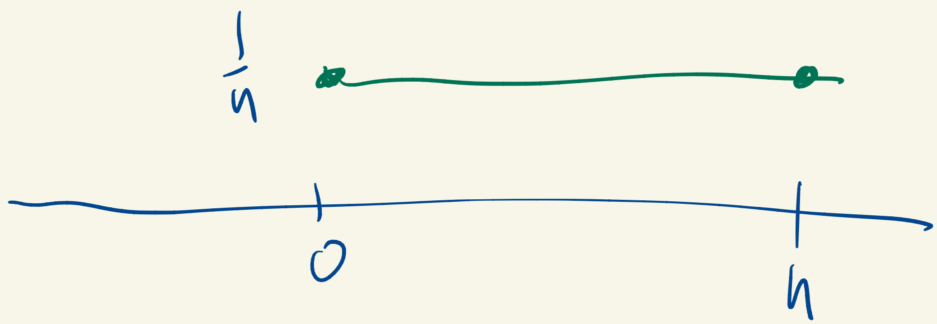
(f_n) $\quad f_n \to 0$ p.w.

1) $f_n = \chi_{[n,\infty)}$ $\quad \int f_n = \infty \ \forall n. \not\to \int 0 = 0$

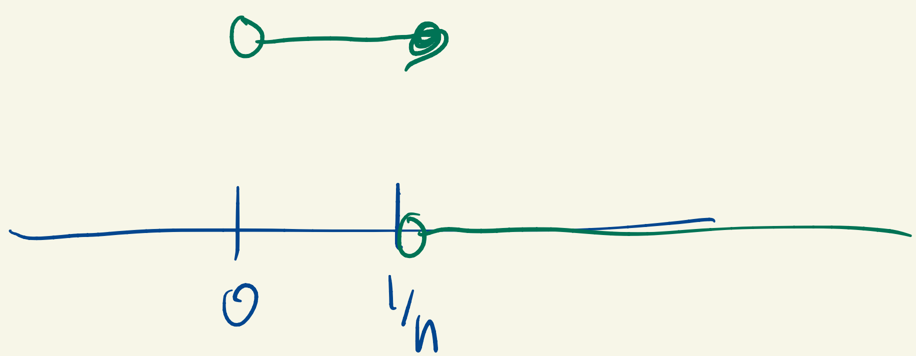
2) $f_n = \chi_{[n,n+1]}$ $\quad \int f_n = 1 \ \forall n \not\to \int 0 = 0$



3) $f_n = \frac{1}{n}\chi_{[0,n]}$ $\quad \int f_n = 1 \neq 0 = \int 0$



4) $f_n = n\chi_{(0,1/n]}$ $\quad \int f_n = 1 \neq 0 = \int 0$



5) $f_n = \begin{cases} \chi_{[n, n+1]} & n \text{ is odd} \\ \chi_{[n, n+3]} & n \text{ is even.} \end{cases}$ $\int f_n = 2 + (-1)^n$

no limit, definitely not 0.

1) - 4) $\int f \leq \lim_{n \to \infty} \int f_n$

5) $\int f \leq \liminf_{n \to \infty} \int f_n$

This holds generally and is known as Fatou's Lemma.

"You can lose area in the limit but you

can't gain area."

Fatou's Lemma: Suppose $f_n \geq 0$ are measurable and $f_n \to f$ p.w. Then $\int f \leq \liminf_{n\to\infty} \int f_n$.

Pf: Let $g_n = \inf_{k\geq n} f_k$. Observe that the g_n's are monotone increasing. Moreover, pointwise

$$\lim_{n\to\infty} g_n = \lim_{n\to\infty} \inf_{k\geq n} f_k = \liminf_{n\to\infty} f_n = f.$$

By the MCT $\lim_{n\to\infty} \int g_n = \int f$.

Note that for each n $g_n \leq f_n$. Thus

$$\liminf_{n\to\infty} \int f_n \geq \liminf_{n\to\infty} \int g_n = \int f. \qquad \square$$