Integration of non-negative measurable functions

$f \geq 0$, measurable $\quad f : \mathbb{R} \to \overline{\mathbb{R}}$

φ is simple, meas, $\quad f \geq \varphi$

$$\int f \geq I(\varphi)$$

$$\boxed{\int f = \underline{\sup \{ I(\varphi) : \varphi \text{ is simple, int, } 0 \leq \varphi \leq f \}}}$$

Exercise: If $\varphi \geq 0$ is simple, int, then

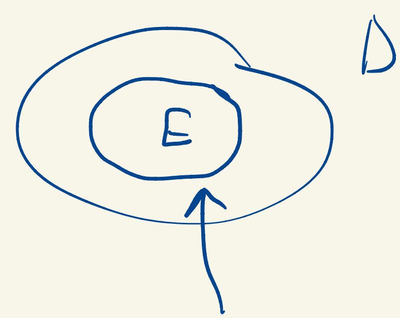
$$\int \varphi = I(\varphi)$$

$f \geq 0$, measurable $\quad f: D \to \overline{\mathbb{R}}$

$$\hat{f} = \begin{cases} f & \text{on } D \\ 0 & \text{on } D^c \end{cases} \qquad \int_D f := \int \hat{f}$$

$f: D \to \overline{\mathbb{R}} \quad f \geq 0$, measurable

$E \subseteq D$, measurable



$$\int_E f = \int_D \chi_E f = \int \widehat{\chi_E f}$$

$$= \int \chi_E \hat{f}$$

Exercise: $\int_D f = \sup \left\{ \int \varphi : 0 \leq \varphi \leq f,\ \varphi \text{ is simple, int, } \varphi = 0 \text{ on } D^c \right\}$

Goal: If $f \geq 0$, $g \geq 0$ are measurable

- $\int \alpha f = \alpha \int f \quad \forall\ \alpha \geq 0$] Easy. Exercise.
- $\int f+g = \int f + \int g$ ⇐ Tricky! Hang tight.
- If $f \leq g$ $\int f \leq \int g$] Super easy.

Suppose $f \geq 0$, measurable

$$\int f = 0 \qquad \text{Is } f = 0 \text{ a.e.?}$$

Tchebyshev's Inequality

If $f \geq 0$ is measurable then for all $\alpha \geq 0$

$$\int f \geq \alpha m(\{f \geq \alpha\}).$$

Pf: Let $\alpha \geq 0$. Let $E = \{f \geq \alpha\}$. Observe $f \geq \alpha \chi_E$.

If $m(E) < \infty$ then

$$\int f \geq \int \alpha \chi_E = \alpha \int \chi_E = \alpha m(E) = \alpha m(\{f \geq \alpha\})$$

If $m(E) = \infty$ then let $E_n = E \cap [-n, n]$ and observe

$$\int f \geq \int \alpha \chi_E \geq \int \alpha \chi_{E_n} = \alpha m(E_n).$$

This is true for all n and $\lim_{n \to \infty} m(E_n) = m(\bigcup E_n) = m(E).$

Similarly $\lim_{n\to\infty} \alpha\, m(E_n) = \alpha\, m(E)$.

$$\int f = 0 \quad f \geq 0 \quad \Rightarrow \quad f = 0 \text{ a.e.}$$

$$E_n = \{ f \geq \tfrac{1}{n} \} \qquad \bigcup E_n = \{ f \neq 0 \}$$

$$\boxed{\int f \geq \frac{1}{n}\, m\left(\{ f \geq \tfrac{1}{n} \}\right)} = \frac{1}{n} \cdot m(E_n)$$

$$n \int f \geq m(E_n) \Rightarrow m(E_n) = 0$$

f, meas, ≥ 0

$$\int f < \infty \Rightarrow f \text{ is finite a.e.}$$

$E_n = \{ f \geq n \}$

$$\int f \geq n \, m(E_n)$$ Tchebyshev!

$$\frac{1}{n} \int f \geq m(E_n) \Rightarrow \lim_{n \to \infty} m(E_n) = 0 \quad \left(\int f \neq \infty \right)$$

$E = \bigcap E_n = \{ f = \infty \}$. Each E_n has finite measure,

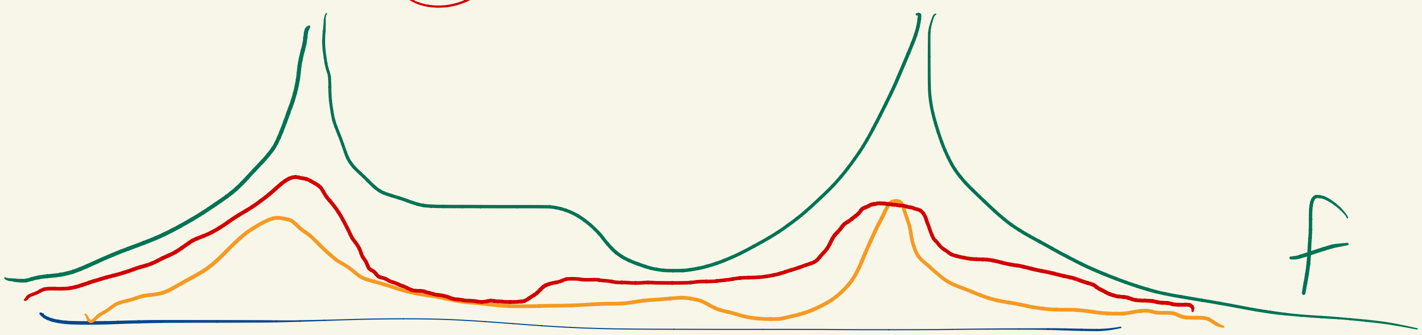
Continuity from above implies $m(E) = \lim m(E_n) = 0$.

Thm Monotone Convergence Theorem.

If $\{f_n\}$ is an increasing sequence of measurable non-negative functions and $f = \lim_{n\to\infty} f_n$ (p.w.)

then

$$\int f = \lim_{n\to\infty} \int f_n .$$



Lemma: Suppose φ is simple and integrable and that

$\{E_k\}$ is an increasing sequence of nested measurable sets. Let $E = \bigcup E_k$. Then

$$\int_E \varphi = \lim_{n\to\infty} \int_{E_n} \varphi .$$

(If $\varphi = 1$, this would be continuity from below)

Pf: We can write $\varphi = \sum_{j=1}^{m} a_j \chi_{F_j}$ with $a_j \neq 0$ and $m(F_j) < \infty$ for all j. Then

$$\int_{E_n} \varphi = \int \chi_{E_n} \varphi = \int \sum_{j=1}^{m} a_j \chi_{E_n} \chi_{F_j}$$

$$= \int \sum_{j=1}^{m} a_j \chi_{E_n \cap F_j}$$

$$= \sum_{j=1}^{m} a_j \, m(E_n \cap F_j).$$

Each $E_n \cap F_j$ is increasing in n for fixed j and $\bigcup (E_n \cap F_j) = \left(\bigcup E_n\right) \cap F_j = E \cap F_j$

Hence $$\lim_{n\to\infty} \int_{E_n} \varphi = \sum_{j=1}^{m} a_j \, m(E \cap F_j) = \int \chi_E \varphi = \int_E \varphi.$$