Lemma: Suppose E is measurable, $m(E) < \infty$. Let $\varepsilon > 0$. There is a continuous function φ such that

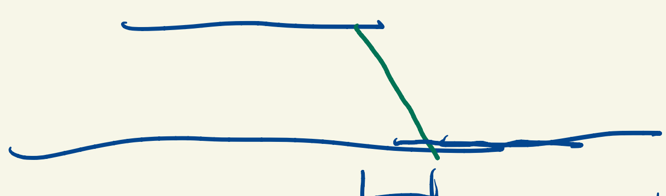
$$m(\chi_E \neq \varphi) < \varepsilon.$$

Pf: Let A be a finite union of intervals such that $m(A \,\Delta\, E) < \varepsilon/2$.

Let $\psi = \chi_A$, so $m(\psi \neq \chi_E) < \varepsilon/2$.

Let φ be a continuous function such that

$m(\varphi \neq \psi) < \varepsilon/2$. Then φ suffices.



as small as you want

Thm (Borel) $\qquad$ $m(f=\pm\infty)=0$

Suppose $f:[a,b]\to\overline{\mathbb{R}}$ is measurable and finite a.e.
Given $\varepsilon>0$ there is a continuous function g with $m(|f-g|>\varepsilon)<\varepsilon$.

Pf: Because $m(|f|=\infty)=0$, continuity from above implies there exists K with $m(|f|\geq K)<\frac{\varepsilon}{2}$.

Let $f_K=\max(\min(f,K),-K)$.

Let $\varphi=\sum_{n=1}^{N} a_n\chi_{E_n}$ be a simple function with $|\varphi-f_K|<\varepsilon$ on $[a,b]$.

$$\{|f| = \infty\} = \bigcap_{n=1}^{\infty} \{|f| \geq n\}$$

$$\underbrace{m(\{|f| = \infty\})}_{0} = \lim_{n\to\infty} m(\{|f| > n\})$$

For each n let g_n be a continuous function that equals χ_{E_n} except on a set of measure no more then $\varepsilon/2N$.

Then $g = \sum_{n=1}^{N} a_n g_n$ satisfies $m(\{g \neq \varphi\}) < \frac{\varepsilon}{2}$.

Now.

$$\{|g-f| > \varepsilon\} \subseteq \{|f| \geq k\} \cup \bigcup_{n=1}^{N} \{g_n \neq \chi_{E_n}\}$$

and $m(\{|g-f| > \varepsilon) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Integration:

A simple function $\varphi : \mathbb{R} \to \mathbb{R}$ is <u>integrable</u> if

$$m(\varphi \neq 0) < \infty.$$

Note $\varphi^{-1}(\{0\}) \neq \emptyset$

$$\varphi = \sum_{k=0}^{n} a_k \chi_{E_k} \qquad E_k = \varphi^{-1}(\{a_k\})$$

$$a_0 = 0$$

$$\underbrace{\{a_0, a_1, \dots, a_n\}}_{\text{disjoint}}$$

Def: $I(\varphi) = \sum_{k=0}^{n} a_k m(E_k)$ $\qquad 0 \cdot \infty = 0$

Exercise: $\underbrace{\text{Integrable}}$ simple functions form a vector space.

$\hookrightarrow m(f \neq 0) < \infty$

Goal: I is linear on the integrable simple functions.

Lemma: If $\varphi = \sum_{k=1}^{n} b_k \chi_{E_k}$ where each E_k is measurable, $m(E_k) < \infty$ and the sets E_k are disjoint then

$$I(\varphi) = \sum_{k=1}^{n} b_k m(E_k).$$

Pf: Observe first that φ is simple and integrable, so $I(\varphi)$

is defined. Without loss of generality we can assume that $b_k = 0$ for some k and $\bigcup E_k = \mathbb{R}$.

Then

$$I(\varphi) = \sum_{a \in \mathbb{R}} a\, m(\{\varphi = a\}).$$

For any $a \in \mathbb{R}$

$$m(\{\varphi = a\}) = m\Big(\bigcup_{b_k = a} E_k\Big)$$

disjoint!

$$= \sum_{b_k = a} m(E_k).$$

Hence $I(\varphi) = \sum_{a \in \mathbb{R}} \sum_{b_k = a} b_k\, m(E_k) = \sum_{k=1}^{n} b_k\, m(E_k)$

Prop: If φ and ψ are simple and integrable then

$$I(c\varphi) = c\, I(\varphi) \quad \text{and}$$

$$I(\varphi + \psi) = I(\varphi) + I(\psi).$$

(I is linear!)

Pf: Scalar multiplication is an exercise.

Let $\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}$ and $\psi = \sum_{j=1}^{m} b_j \chi_{F_j}$

in standard form.

Let $A_{ij} = E_i \cap F_j$. Observe that the sets

A_{ij} are disjoint. Moreover

$$E_i = \bigcup_{j=1}^{m} A_{ij}$$

$$F_j = \bigcup_{i=1}^{n} A_{ij} .$$

Then $I(\varphi + \psi) = \sum_{i=1}^{n} \sum_{j=1}^{m} (a_i + b_j)\, m(A_{ij})$

since $\varphi + \psi = \sum_{i=1}^{n} \sum_{j=1}^{m} (a_i + b_j)\, \chi_{A_{ij}}$.

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} a_i\, m(A_{ij}) + \sum_{j=1}^{m} \sum_{i=1}^{n} b_j\, m(A_{ij})$$

$$= \sum_{i=1}^{n} a_i \left(\sum_{j=1}^{m} m(A_{ij}) \right) + \sum_{j=1}^{m} b_j \left(\sum_{i=1}^{n} m(A_{ij}) \right)$$

$$= \sum_{i=1}^{n} a_i\, m(E_i) + \sum_{j=1}^{m} b_j\, m(F_j)$$

$$= I(\varphi) + I(\psi).$$

Cor: If E_i $i = 1, \dots, n$ are measurable sets with finite measure then $\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}$ is simple and measurable and

$$I(\varphi) = \sum_{i=1}^{n} a_i m(E_i).$$

Pf: This is a consequence of linearity once we establish $I(\chi_{E_i}) = m(E_i)$.

The standard neprepsentation of

$$\chi_{E_i} = 0 \cdot \chi_{E_i^c} + 1 \cdot \chi_{E_i}$$

and hence, by definition,

$$I(\chi_{E_i}) = 0 \cdot m(E_i^c) + 1 \cdot m(E_i)$$
$$= m(E_i).$$

Lemma: If φ is integrable and simple and $\varphi \geq 0$ a.e. then $I(\varphi) \geq 0$.

Pf: $\varphi = \sum_{k=0}^{n} a_k \chi_{E_k}$ where whenever $a_k < 0$, $m(E_k) = 0$,

Thus $I(\varphi) = \sum_{k=0}^{n} a_k \, m(E_k) \geq 0.$

Cor: If φ and ψ are simple and integrable and

$\varphi \leq \psi$ a.e. then

$$I(\varphi) \leq I(\psi).$$

Pf: Observe $\psi - \varphi \geq 0$ a.e.

So by the lemma

$$I(\psi - \varphi) \geq 0.$$

But by linearity $I(\psi - \varphi) = I(\psi) - I(\varphi)$ so

$$I(\psi) \geq I(\varphi).$$