Basic construction

Def: A function $f: \mathbb{R} \to \mathbb{R}$ (or $D \to \mathbb{R}$ with D measurable) is a <u>simple function</u> if it is measurable and attains finitely many values.

$$f(\mathbb{R}) = \{a_0, a_1, a_2, \dots, a_n\}$$

$a_0 = 0$

measurable

$$E_k = f^{-1}(\{a_k\})$$

$$f = \sum_{k=0}^{n} a_k \chi_{E_k}$$

$$= \sum_{k=1}^{n} a_k \chi_{E_k}$$

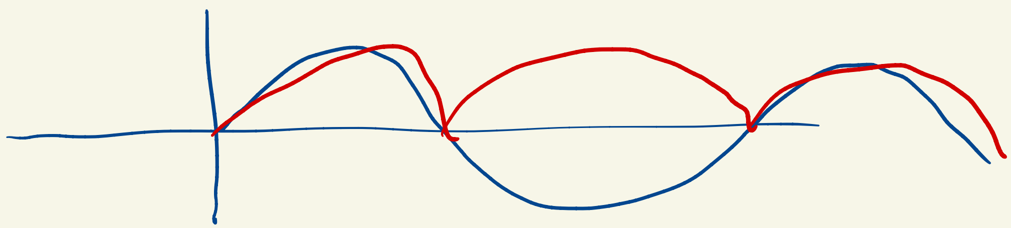
standard form

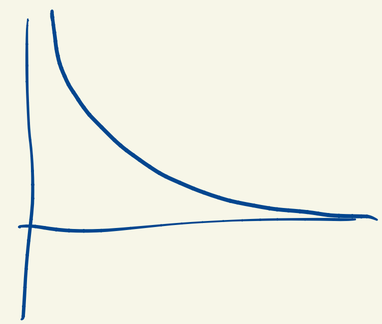
$$\sum_{k=0}^{n} a_k m(E_k) = \sum_{k=1}^{n} a_k m(E_k)$$

Goal: Given a measurable function f there is a sequence ϕ_n of simple functions such that

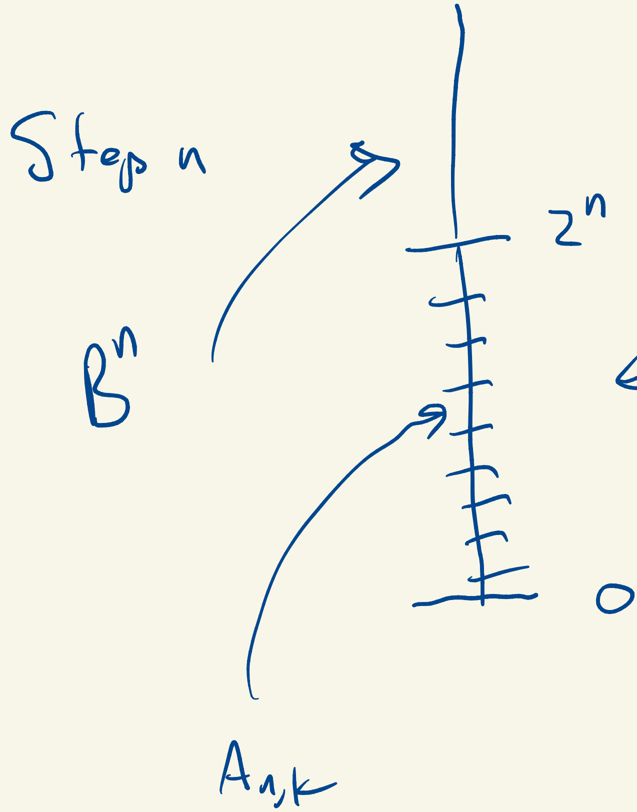
$$0 \leq |\phi_1| \leq |\phi_2| \leq \cdots \quad \text{and where} \quad \phi_n \to f \text{ pointwise}$$

(and uniformly on any set where f is bounded).





Case 1) $f \geq 0$



split into pieces of size $\frac{1}{2^n}$

2^{2n} pieces $\underbrace{\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)}$

$A_{n,k}$ $\qquad 0 \leq k \leq 2^{2n}-1$

$B_n = [2^n, \infty)$

$E_{n,k} = f^{-1}(A_{n,k})$ (measurable)

$F_n = f^{-1}(B_n)$

$$\varphi_n = \left[\sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} \chi_{E_{n,k}} \right] + 2^n \chi_{F_n}$$

$$x \in E_{n,k} = f^{-1}(A_{n,k})$$

C

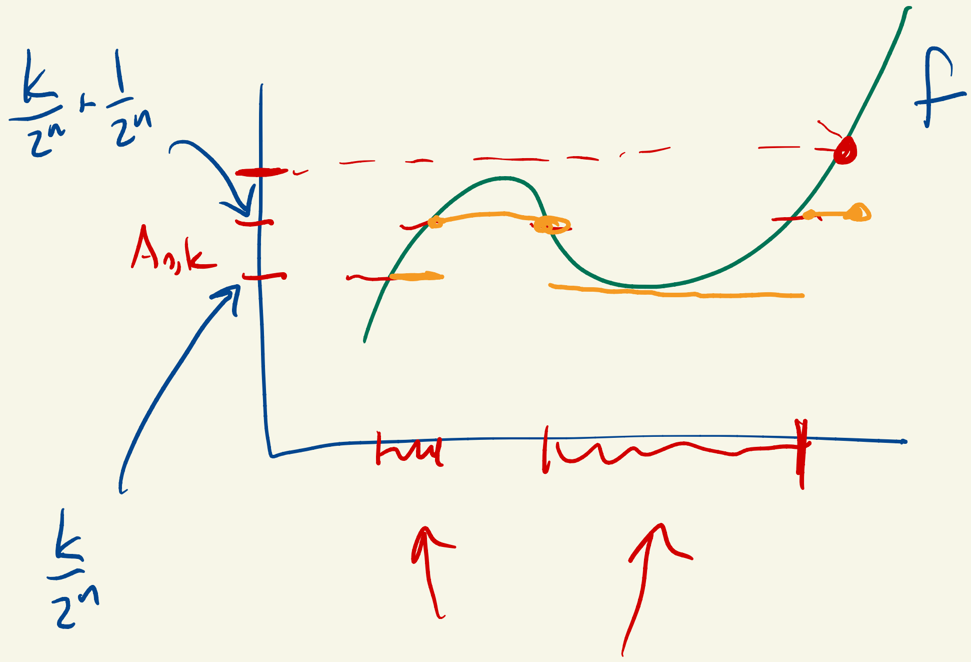
$$f(x) \in A_{n,k}$$

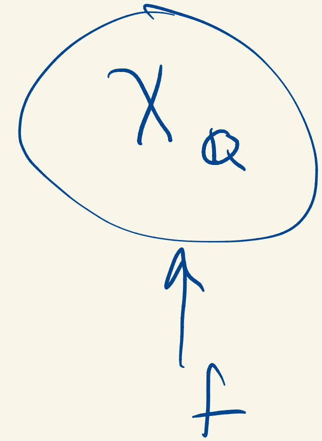
$$\frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n} = \frac{k}{2^n} + \frac{1}{2^n}$$

$$\varphi_n(x) = \frac{k}{2^n}$$

$$|f(x) - \varphi_n(x)| < \frac{1}{2^n}$$

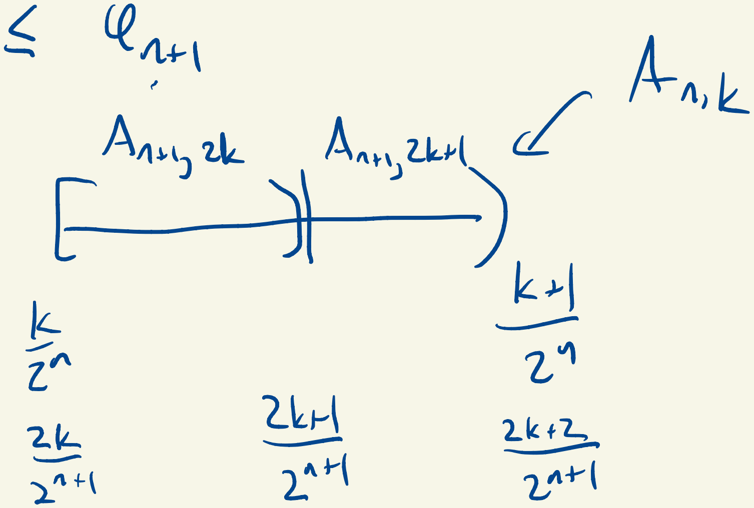
$$E_n = \bigcup_k E_{n,k} = f^{-1}([0, 2^n)) \quad \text{on } E_n \quad |f - \varphi_n| < \frac{1}{2^n}$$





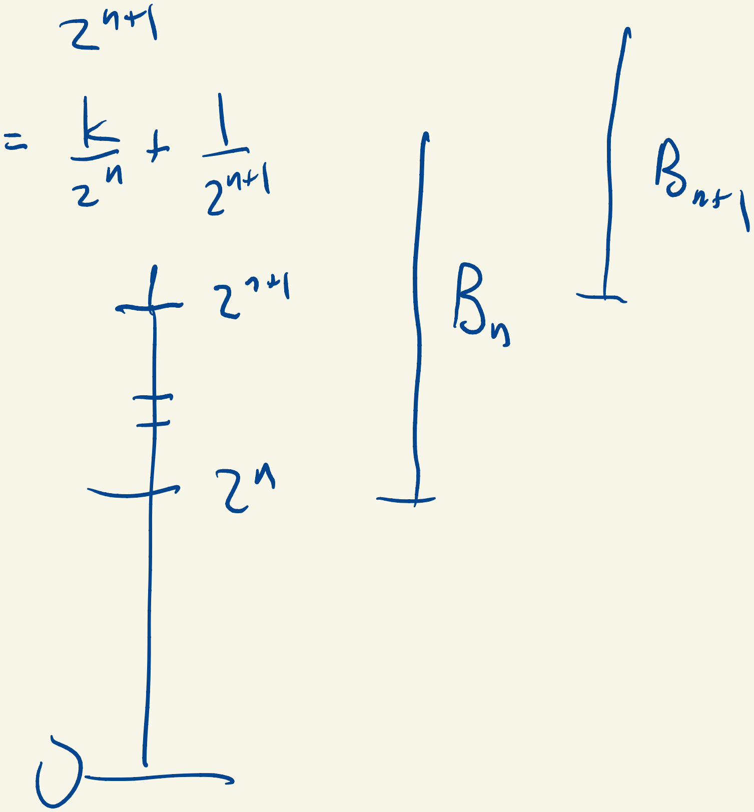
$$f^{-1}(A_{n,k}) = E_{n,k}$$

Claim: $\varphi_n \leq \varphi_{n+1}$



On $E_{n,k}$ $\quad \varphi_n(x) = \frac{k}{2^n}$, $\quad \varphi_{n+1}(x) = \begin{cases} k/2^n \\ k/2^n + \frac{1}{2^{n+1}} \end{cases}$

$$E_{n,k} = \underbrace{E_{n+1,2k}}_{\varphi_{n+1} = \frac{2k}{2^{n+1}} = \frac{k}{2^n}} \cup \underbrace{E_{n+1,2k+1}}_{\varphi_{n+1} = \frac{2k+1}{2^{n+1}} = \frac{k}{2^n} + \frac{1}{2^{n+1}}}$$



So $\quad \varphi_{n+1}(x) \geq \varphi_n(x)$

If $x \notin E_n = \bigcup_k E_{n,k}$ then $\varphi_n(x) = 2^n$, $\varphi_{n+1}(x) \geq 2^n$

Claim: $\varphi_n \rightrightarrows f$ on any set where f is bounded.

If F is bounded on H with $|f| \leq K$

pick N so that $2^N > K$. Then if $n \geq N$

$$|\varphi_n - f| < \frac{1}{2^n} \text{ on } H.$$

So $\varphi_n \rightrightarrows f$ on H.

Claim If $f(x) = \infty$ $\varphi_n(x) \to \infty = f(x)$.

$$\varphi_n(x) = 2^n \to \infty.$$

Theorem: Suppose $f : D \to \overline{\mathbb{R}}$ is measurable and non-negative. Then there exists a sequence of simple functions φ_n with

$$0 \le \varphi_1 \le \varphi_2 \le \cdots$$

and $\varphi_n \le f$ and

$$\varphi_n \to f \quad \text{pointwise}$$

$$\varphi_n \rightrightarrows f \quad \text{on any set where } f \text{ is bounded.}$$