$$\left.\begin{array}{l} f^{-1}((a,\infty)) \in \mathcal{M} \\ f^{-1}((a,b)) \in \mathcal{M} \end{array}\right]$$

f is measurable $\Leftrightarrow$ $f^{-1}(B) \in \mathcal{M}$ $\forall$ borel sets B.

Examples:

1) continuous functions

$$f^{-1}(\text{open}) = \text{open}$$

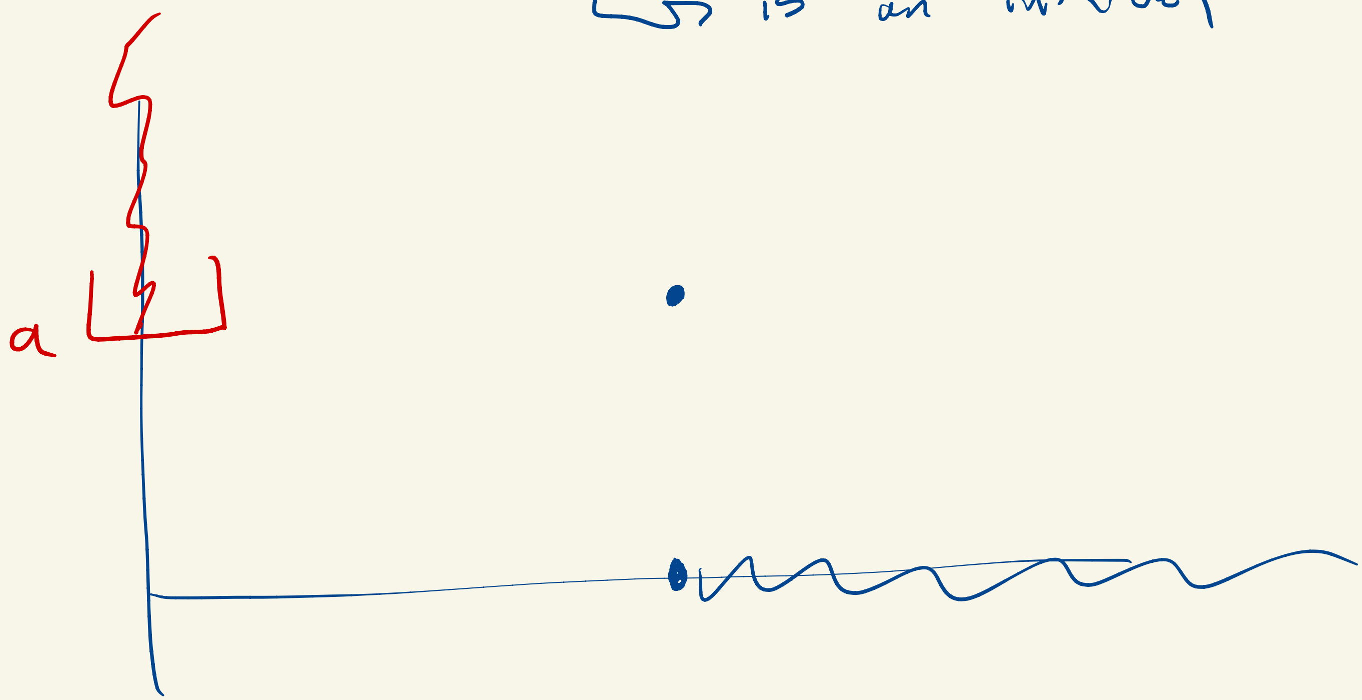
2) step functions $[a,b]$

$$f^{-1}((a,\infty)) \to \text{finite union of intervals}$$

3) monotone functions $\mathbb{R} \to \mathbb{R}$ are measurable.

$$f^{-1}([a, \infty))$$

$\hookrightarrow$ is an interval



upper semi-continuous $f^{-1}((a,\infty))$ is open

If $N \subseteq \mathbb{R}$ is null

and $f: N \to \mathbb{R}$

then f is measurable.

$f^{-1}(A) \subseteq N$ and is meas.

If f is meas and $\underbrace{g = f \text{ except on a null}}$

set N then g is measurable.

$\{f > a\}$ vs $\{g > a\}$ $f = g$ almost everywhere.

E_f E_g

measurable

measurable?

$$E_f \,\Delta\, E_g \subseteq N$$

Exercise: If $f : D \to \mathbb{R}$ is measurable and $E \subseteq D$ is measurable then $f|_E$ is measurable.

Thm: The measurable real-valued functions on a measurable set $D \subseteq \mathbb{R}$ form a vector space and moreover an algebra.

i.e. if $f, g : D \to \mathbb{R}$ are measurable then so are:

1) cf is measurable if $c \in \mathbb{R}$

2) $f+g$ is measurable

3) $f \cdot g$ is measurable

Exercise: 1)

Hard work: 2)

Want $\{f+g > \alpha\}$ to be measurable.

$$f(x) + g(x) > \alpha \iff f(x) > \alpha - g(x)$$

$$\Longleftrightarrow \exists\, r \in \mathbb{Q}$$

$$f(x) > r > \alpha - g(x)$$

$$\{f+g>\alpha\} = \bigcup_{r\in\mathbb{Q}} \{f>r\} \cap \underbrace{\{r>\alpha-g\}}_{\{g>\alpha-r\}}$$

So $\{f+g>\alpha\}$ is a countable union of measurable sets and is measurable.

To tackle $f \cdot g$ is measurable first start with $f \cdot f = f^2$ is measurable.

$$\{f^2 > \alpha\} = \underbrace{\{f > \sqrt{\alpha}\}}_{\text{meas}} \cup \underbrace{\{f < -\sqrt{\alpha}\}}_{\text{meas.}}$$

$$(f+g)^2 = f^2 + g^2 + 2fg$$

$$fg = \frac{1}{2}\left[(f+g)^2 - f^2 - g^2\right]$$

Given f, g measurable,

$h = \max(f, g)$ is measurable.

$$\{h > \alpha\} = \underbrace{\underset{\text{meas}}{\{f > \alpha\}} \cup \underset{\text{meas}}{\{g > \alpha\}}}_{\text{meas}}$$

$f_1, f_2, \ldots, f_n$ all meas

$f = \max(f_1, f_2, \ldots, f_n)$ is meas.

What about $f_1, f_2, f_3, \ldots$

$$f = \sup_n f_n \qquad f_k(x) = k$$

We'll work with extended real-valued functions

$$f : D \to \overline{\mathbb{R}} \to \mathbb{R} \cup \{\infty, -\infty\}$$

Such an f is meas if

$$f^{-1}\big((a, \infty]\big) \text{ are measurable.}$$

$f + g$ need not be defined for such functions.

$f(x) = \infty \qquad g(x) = -\infty$

$0 \cdot \infty$

Note: in most cases we work with,
the functions will be finite
almost everywhere

$f = \sup f_n \qquad$ each f_n is meas.

$$\{ f > \alpha \} = \bigcup_n \{ f_n > \alpha \}$$

Exercise: verify the above equality.

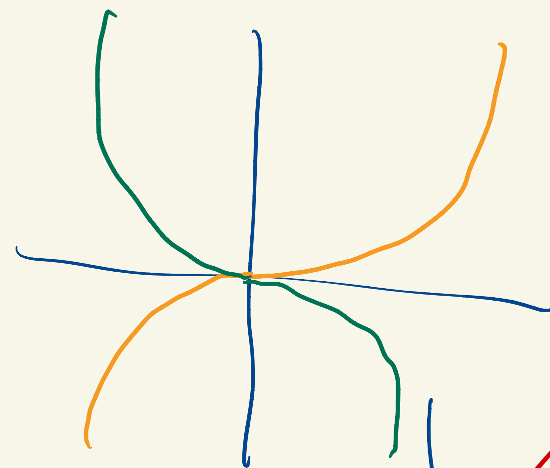
$\inf f_k = -\sup(-f_k)$ is meas if each f_k is.

Given a function $f : D \subseteq \mathbb{R} \to \mathbb{R}$

$$f^+ = \max(f, 0) \qquad f^+ \geq 0$$

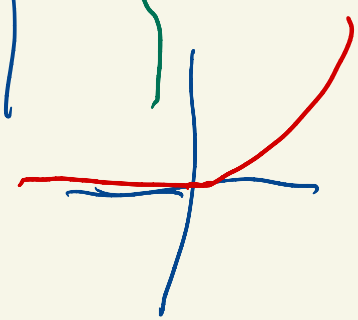
$$f^- = \max(-f, 0) \qquad f^- \geq 0$$

$$f = f^+ - f^-$$



If f is meas, so are f^+, f^-.

If f^+ and f^- are meas, so is f.



$$|f| = f^+ + f^-$$

If f is meas, so is $|f|$.

Exercise: Is the converse true?

If f_n is a seq of meas functions

$$\limsup_{n\to\infty} f_n = \underbrace{\inf_{n\geq 1} \underbrace{\sup_{m\geq n} f_m}_{\text{For each } n, \text{ this is meas.}}}_{\text{measurable.}}$$

$\liminf_{n\to\infty} f_n$ is measurable if each f_n is.

If $f_n \to f$ pointwise and each f_n is measurable then f is measurable.

$\boxed{\text{The pointwise limit of measurable functions is measurable}}$

Exercise: If f_n is a sequence of measurable functions that converges pointwise almost everywhere to some f then f is measurable

$\mathbb{R} \setminus N$

$f_n \to f$ on

$f_n|_{\mathbb{R}\setminus N}$ are meas

$\Rightarrow$ $f|_{\mathbb{R}\setminus N}$ is meas.

$f : D \to \mathbb{R}$ is meas

$E \subseteq D$ is meas

$f|_E$ is meas.

$f^{-1}((a,\infty]) \setminus N$ is meas.

$f^{-1}((a,\infty]) \cap N$ is meas

Egoroff. Suppose $D \subseteq \mathbb{R}$ is measurable and $m(D) < \infty$.

If $\{f_n\}$ is a sequence of meas functions $D \to R$ converging pw a.e. to f then given $\varepsilon > 0$ there exists a measurable set $E \subseteq D$ such that $m(D \setminus E) < \varepsilon$ and $f_n \rightrightarrows f$ on E.

Littlewood's 3 Principles

* A measurable set is nearly
  a) an open set
  b) a closed set
  c) a Borel set (G_δ)

* Pointwise a.e. convergence is nearly uniform convergence

* Measurable functions are nearly continuous functions.