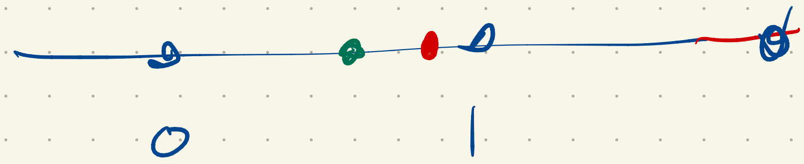
$\dot{+} : [0,1) \to [0,1)$



$$a \dot{+} b = \begin{cases} a+b & a+b<1 \\ a+b-1 & a+b > 1 \end{cases}$$

$H = \mathbb{Q} \cap [0,1)$

$$q_1 \dot{+} z \to \begin{cases} q_1+z \\ q_1+z-1 \end{cases}$$

$H \dot{+} z \qquad z \in [0,1)$

$$q_2 \dot{+} z \to \begin{cases} q_2+z \\ q_2+z-1 \end{cases}$$

$x \sim y$ if $x - y \in \mathbb{Q}$

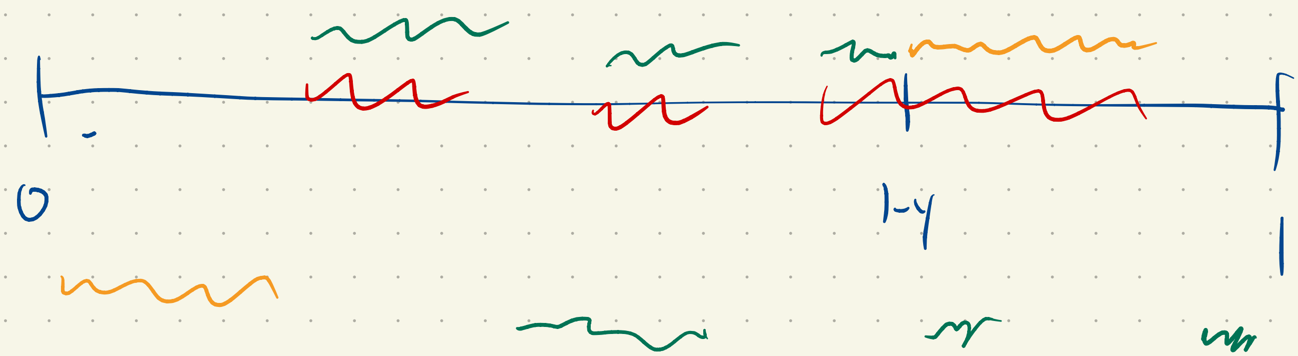
Exercise $x \sim y \Leftrightarrow x, y$ live in the same coset.

How many cosets: uncountably many.

A: choose one element from each coset.

claim: A is not measurable.

$$A \dot{+} y = \Big(\big(A \cap [0, 1-y)\big) + y \Big) \cup \Big(\big(A \cap [1\ldots, 1]\big) - (1-y) \Big)$$



Claim: $r_1, r_2 \in \underbrace{\mathbb{Q} \cap [0,1)}_{H}$

$$A \dot{+} r_1 \cap A \dot{+} r_2 = \emptyset \quad \text{unless} \quad r_1 = r_2$$

$A \cap A \overset{\circ}{+} r = \phi$ unless $r = 0$

$r \in H$

a_1 $\qquad$ $\underbrace{a_2 \overset{\circ}{+} r}$

$\begin{cases} a_2 + r \\ a_2 + r - 1 \end{cases}$

$a_1, a_2 \in A$

$\begin{cases} a_1 = a_2 + r \\ a_1 = a_2 + r - 1 \end{cases}$

$a_1 - a_2 = \begin{cases} r \\ r - 1 \end{cases} \in \mathbb{Q}$

$a_1 \sim a_2 \Rightarrow a_1 = a_2$

$\Rightarrow$ $r = 0$

$r - 1 = 0 \Rightarrow \cancel{r = 1}$

$r \in H \subseteq [0, 1)$

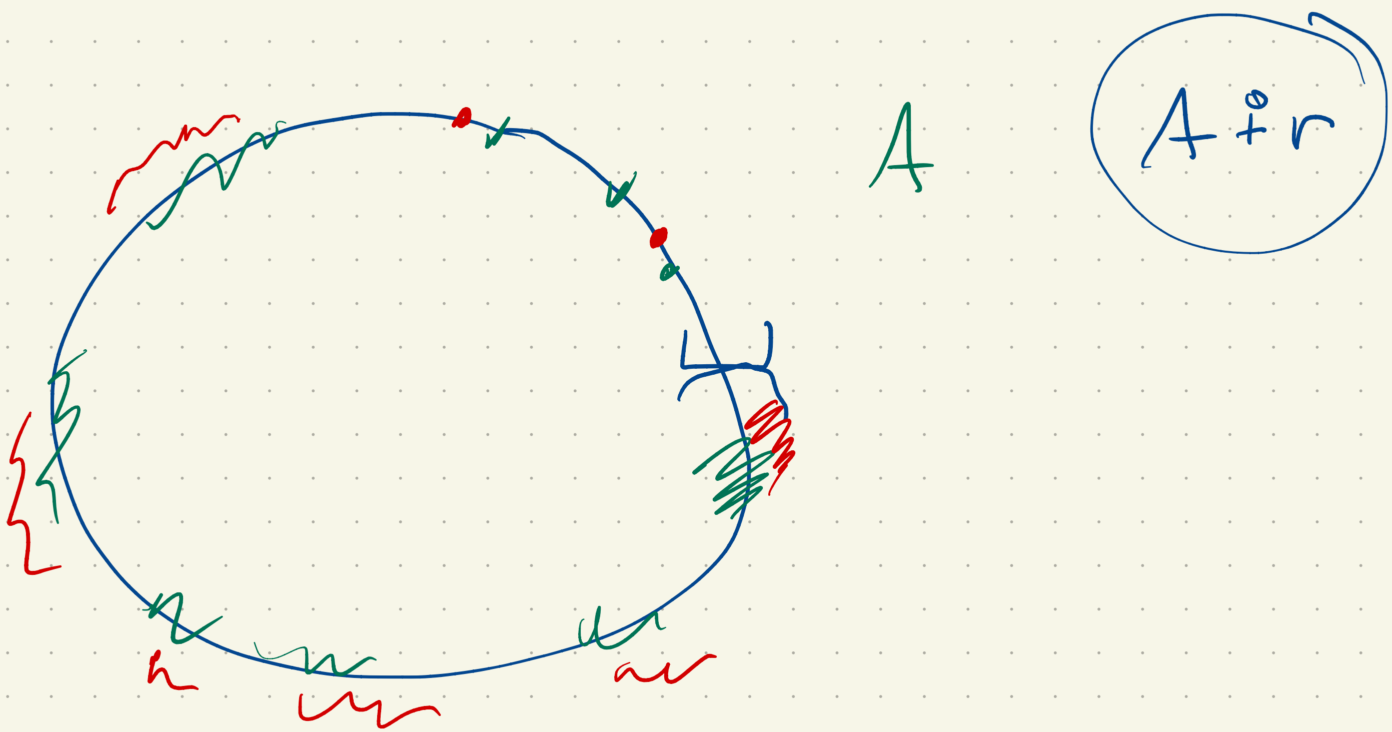
$\Rightarrow r = 0.$

Claim $\bigcup_{r \in H} A \dot{+} r = [0,1)$

Let $y \in [0,1)$. Then $y \sim a$ for some $a \in A$.

$y = a \dot{+} r$ for some $r \in H$.

$\Rightarrow y \in A \dot{+} r$



Suppose we have $\ell : P(\mathbb{R}) \to [0,\infty]$

that is translation invariant and countably additive.

$\Rightarrow$ finite additive

$\Rightarrow$ monotone

Claim: either $\ell([0,1)) = 0$ or $\ell([0,1)) = \infty$

$$\boxed{\ell(A \dot{+} r) = \ell(A)}$$

$$\ell([0,1)) = \sum_{q \in H} \ell(A \dot{+} q) \quad \text{(countable additivity)}$$

$$= \sum_{q \in H} \ell(A)$$

If $\ell(A)=0 \Rightarrow \ell([0,1))=0$

otherwise $\ell([0,1))=\infty$.

A is not measurable.

If it were then $A \oplus r$ would be for any $r \in H$.

$$1 = \sum_{q \in H} m(A \oplus q) = \sum_{q \in H} m(A) \Rightarrow \Leftarrow$$

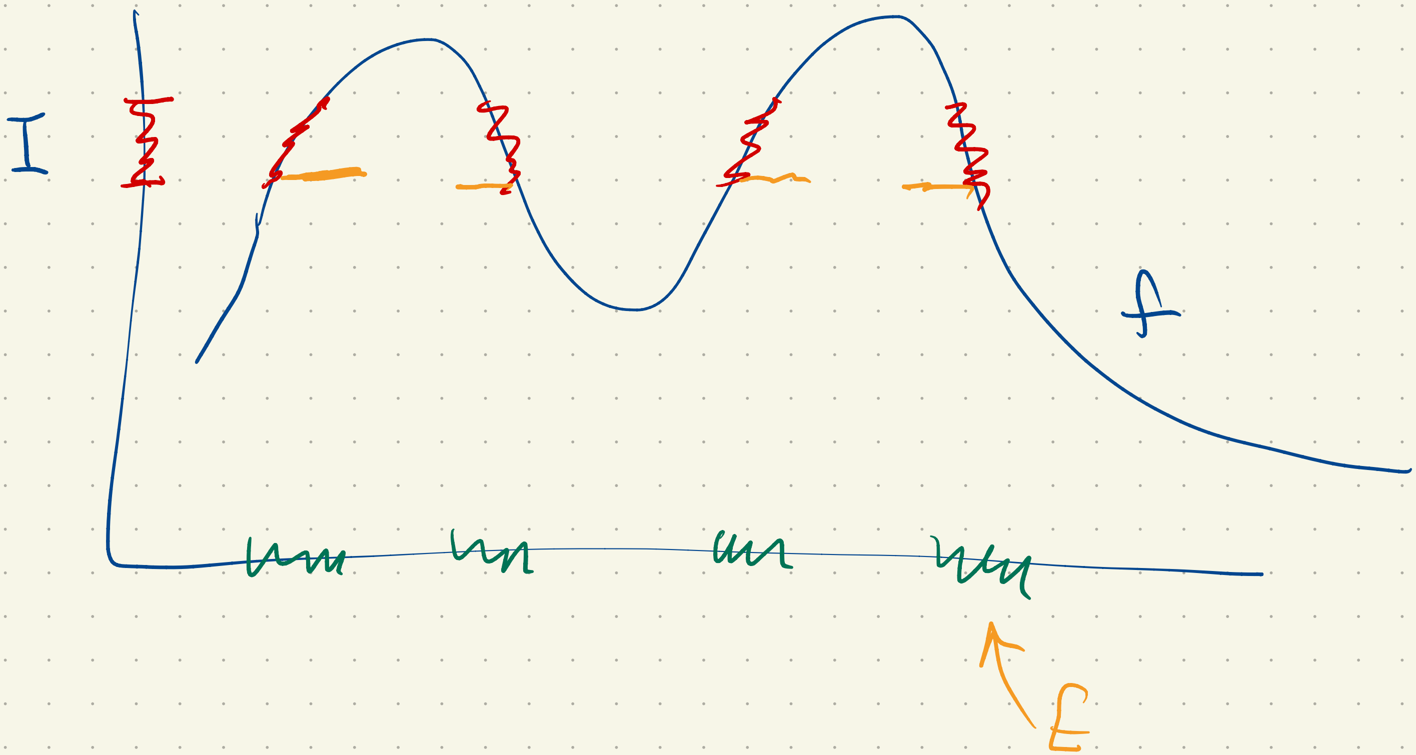
Measurable functions.

Simple functions.

$$f = c_1 \chi_{E_1} + \cdots + c_n \chi_{E_n} \qquad E_k \text{ is measurable.}$$

$$\int f \overset{\text{should}}{=} \sum c_k m(E_k)$$

$$f = \chi_{\mathbb{Q} \cap [0,1]} \qquad \int_0^1 f = 1 \cdot m(\mathbb{Q} \cap [0,1]) = 0.$$



$$E = f^{-1}(I)$$

We'll want $f^{-1}(I)$ is measurable whenever I is an interval.

Def: A function $f:\mathbb{R}\to\mathbb{R}$ is (Lebesgue) <u>measurable</u> if

$\underbrace{f^{-1}((a,\infty))}_{\{f>a\}}$ is measurable for all $a\in\mathbb{R}$.

If $D\subseteq\mathbb{R}$ is measurable, $f:D\to\mathbb{R}$ is measurable if $f^{-1}((a,\infty))$ is measurable for all $a\in\mathbb{R}$.

$$D=\bigcup_{n\in\mathbb{N}} f^{-1}((-n,\infty))$$

Remark: If f is measurable then

$f^{-1}((a,b])$ is measurable for all $a,b\in\mathbb{R}$.

$$f^{-1}((a,\infty)) \setminus f^{-1}((b,\infty))$$

$$\bigcup_{n\in\mathbb{N}} (a, a+n] = (a, \infty)$$

$$f^{-1}((a,\infty)) = \bigcup_{n\in\mathbb{N}} f^{-1}((a,a+n])$$

$$f^{-1}\left(\bigcup_{\alpha\in J} A_\alpha\right) = \bigcup_{\alpha\in J} f^{-1}(A_\alpha)$$

$$f^{-1}\left(\bigcap_{\alpha\in J} A_\alpha\right) = \bigcap_{\alpha\in J} f^{-1}(A_\alpha)$$

$$f^{-1}(A^c) = \left(f^{-1}(A)\right)^c$$

Exercise: f is measurable iff

$$f^{-1}((a,b)) \text{ are measurable } \forall a,b \in \mathbb{R}.$$

$$(a,b) = \bigcup_{n\in\mathbb{N}} (a, b-\tfrac{1}{n}]$$

$$f^{-1}((a,b)) = f^{-1}\Big(\bigcup_{n\in\mathbb{N}} (a, b-\tfrac{1}{n}]\Big)$$

$$= \bigcup_{n\in\mathbb{N}} f^{-1}(a, b-\tfrac{1}{n}]$$

$f^{-1}(\{a\})$ is measurable. $f^{-1}\Big(\bigcap_{n} (a-\tfrac{1}{n}, a+\tfrac{1}{n})\Big)$

Exercise: If $\mathcal{A}$ is a σ-algebra of subsets of X

$$f : X \to Y$$

then $\{V : f^{-1}(V) \in \mathcal{A}\}$ is itself

a σ-algebra of subsets of Y.

$\mathcal{M}$ is a σ-alg of subsets of $\mathbb{R}$,

$f : \mathbb{R} \to \mathbb{R}$ $\quad\{V : f^{-1}(V) \in \mathcal{M}\}$ is

a σ-alg.

$$f^{-1}((a,\infty)) \in \mathcal{M}$$

$$f^{-1}((a,b)) \in \mathcal{M}$$

f is measurable $\Leftrightarrow$ $f^{-1}(B) \in \mathcal{M}$ $\forall$ borel-sets B.

Examples: 1) continuous functions

$$f^{-1}(\text{open}) = \text{open}$$

2) step functions $[a,b]$

$f^{-1}((a,\infty)) \to$ finite union of intervals