$E \subseteq \mathbb{R}$ is measurable

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$

If $A \subseteq \mathbb{R}$ is any set I can find G, ~~open~~ G_δ $G \supseteq A$

$$m^*(G) = m^*(A)$$

$\forall n$ find an open set $U_n \supseteq A$ $\quad m^*(U_n) \leq m^*(A) + \frac{1}{n}$

$\{I_k\}$ $\quad \sum_k \ell(I_k) \leq m^*(A) + \frac{1}{n}$

$A \subseteq \bigcup I_k \quad U_n = \bigcup I_k \quad m^*(U_n) \leq \sum m^*(I_k) \leq m^*(A) + \frac{1}{n}$

$G = \bigcap U_n \qquad G \supseteq A$

$$m^*(A) \le m^*(G) \le m^*(U_n) \le m^*(A) + \frac{1}{n}$$

$\forall n$

$$m^*(A) = m^*(G)$$

$G \supseteq A$

$m^*(G) = m^*(A)$

$m^*(G \setminus A) \overset{?}{=} 0$

$G \quad G_\delta \Rightarrow$ measurable

$m^*(G \setminus A) = 0 \Rightarrow G \setminus A$ is meas.

$A = G \setminus (G \setminus A)$

$G \cap (G \setminus A)^c$

$G \cap (G \cap A^c)^c$

$G \cap (G^c \cup A) = A$

Thm: TFAE

1) E is measurable

2) $\forall \varepsilon > 0$ there exists an open set U with $U \supseteq E$

and $m^*(U \setminus E) < \varepsilon$

3) There exists a G_δ set $G \supseteq E$ and $m^*(G \setminus E) = 0$.

Pf: We just proved $3) \Rightarrow 1)$.

$2) \Rightarrow 3)$

$\forall n$ pick an open set U_n such that $m^*(U_n \setminus E) < \frac{1}{n}$

and $E \subseteq U_n$.

Let $G = \bigcap U_n$. Then $E \subseteq G$ and by monotonicity

$$m^*(G \setminus E) \leq m^*(O_n \setminus E) < \frac{1}{n} \quad \forall n. \quad \text{Hence } m^*(G \setminus E) = 0.$$

1) $\Rightarrow$ 2)

Case 1) $m^*(E) < \infty$.

Let $\varepsilon > 0$. Let $\{I_k\}$ be a measuring cover of E such that $\sum \ell(I_k) < m^*(E) + \varepsilon$.

Let $U = \bigcup I_k$. Then $E \subset U$ and

$$m^*(U) \leq \sum \ell(I_k) < m^*(E) + \varepsilon$$

But because E is measurable

$$m^*(U) = m^*(U \cap E) + m^*(U \cap E^c)$$

$$= m^*(E) + m^*(U \cap E^c).$$

Here $m^*(E) + m^*(U \setminus E) < m^*(E) + \varepsilon$.

Since $m^*(E) < \infty$, $m^*(U \setminus E) < \varepsilon$.

Case 2) $m^*(E) = \infty$.

For each $n \in \mathbb{N}$ let $E_n = E \cap [-n, n]$.

Let $\varepsilon > 0$. Find open sets U_n such that each $U_n \supseteq E_n$

and $m^*(U_n \setminus E_n) < \frac{\varepsilon}{2^n}$.

Let $U = \bigcup U_n$. Since each $U_n \supseteq E_n$ $U \supseteq E$.

Moreover $m^*(U_n \setminus E) \leq m^*(U_n \setminus E_n) < \frac{\varepsilon}{2^n}$ for all n.

Additionally $\bigcup (U_n \setminus E) = \bigcup (U_n \cap E^c)$

$$= \left(\bigcup U_n\right) \cap E^c$$

$$= U \setminus E.$$

Thus $m^*(U \setminus E) \leq \sum_{n=1}^{\infty} m^*(U_n \setminus E)$

$$\leq \sum_{n=1}^{\infty} m^*(U_n \setminus E_n)$$

$$< \varepsilon.$$

Upshot: Borel sets $\quad \mathbb{R}$

σ-algebra, contains the open sets

smallest such σ-algebra.

A measurable set is almost a Borel set. $E \cup N = G$

Exercise: TFAE

1) E is measurable
2) $\forall \, \varepsilon > 0 \;\exists$ a closed set F with $m^*(E \setminus F) < \varepsilon$
3) $\exists$ a F_σ set $F \subseteq E$ with $m^*(E \setminus F) = 0$.

A measurable set is almost a Borel set

$$E = F \cup N.$$

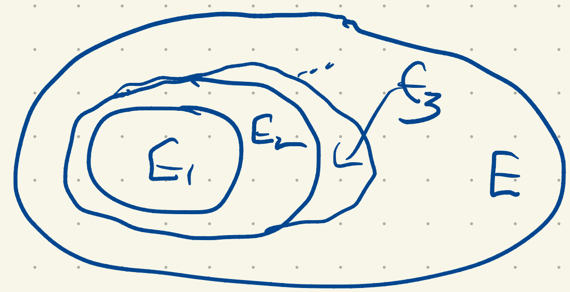
Exercise: E is measurable $\Leftrightarrow \forall \varepsilon > 0$ there exists an open set U and a closed set F with $F \subseteq E \subseteq U$ and $m^*(U \setminus F) < \varepsilon$.

SLOGAN: Every measurable set is nearly an open set and nearly a closed set.

Lebesgue measure possesses a kind of continuity.

$$E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$$ ← measurable

$$E = \bigcup_{k=1}^{\infty} E_k$$



Want $m(E) = \lim_{k\to\infty} m(E_k)$

$$G_1 = E_1 \qquad \underbrace{G_1 \cup G_2}_{E_1 \cup E_2} \cup G_3 = \bigcup G_k = \bigcup E_k$$

$$G_2 = E_2 \setminus G_1$$

$$G_3 = E_3 \setminus (E_1 \cup E_2)$$

G_k's are disjoint.

$$m\left(\bigcup E_k\right) = m\left(\bigcup G_k\right) = \sum_{k=1}^{\infty} m(G_k)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} m(G_k)$$

$$= \lim_{n\to\infty} m\left(\bigcup_{k=1}^{n} G_k\right)$$

$$= \lim_{n\to\infty} m\left(\bigcup_{k=1}^{n} E_k\right)$$

$$= \lim_{n\to\infty} m(E_n)$$

"Continuity from below"

You can't pick up extra length in the limit.

$\{F_k\}$ F_k measurable

$F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$

$\bigcap F_k$

$$\boxed{m\left(\bigcap F_k\right) = \lim_{k\to\infty} m(F_k) \ ?}$$

$F_n = [n, \infty)$

$m^*(F_n) = \infty$

$\bigcap F_n = \emptyset$

$$\underbrace{\lim_{n\to\infty} m^*(F_n)}_{\infty} = \underbrace{m^*\left(\bigcap F_n\right)}_{0} \ ?$$

Prop (Continuity from above)

Let $\{F_k\}$ be a sequence of measurable sets such that $m(F_1) < \infty$ and

$$F_1 \supseteq F_2 \supseteq F_3 \cdots.$$

Then $m(\bigcap F_k) = \lim_{k\to\infty} m(F_k)$.

See notes. $F_1 \setminus F_k$

A non-measurable set.

$\mathbb{R}$ with addition is a group.

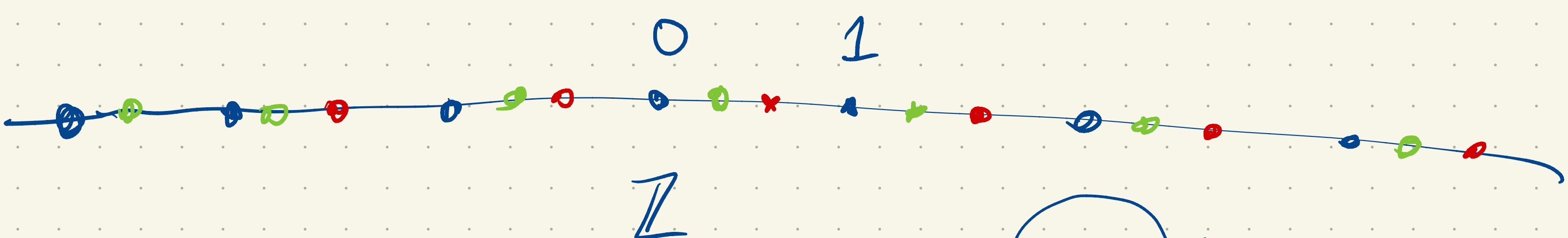
$\mathbb{Q} \subseteq \mathbb{R}$ is a subgroup.

$\mathbb{Q} + z \quad z \in \mathbb{R}$ (cosets)

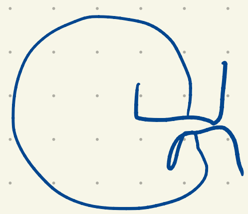
$\mathbb{Z} \subseteq \mathbb{R}$ are a subgroup of $\mathbb{R}$

$\mathbb{Z} \subseteq \mathbb{R}$ is a (normal) subgroup.

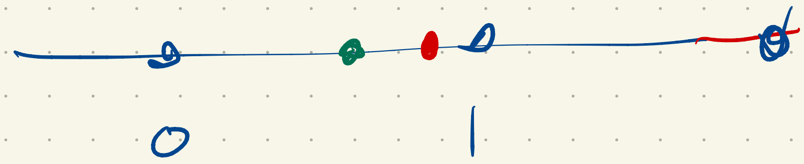
$\mathbb{R}/\mathbb{Z}$ (set of cosets)



$\mathbb{R}/\mathbb{Z} \sim [0,1)$



$\dot{+} : \underline{[0,1)} \to \underline{[0,1)}$



$$a \dot{+} b = \begin{cases} a+b & a+b<1 \\ a+b-1 & a+b>1 \end{cases}$$

$H = \mathbb{Q} \cap [0,1)$

$$q_1 \dot{+} z \to \begin{cases} q_1+z \\ q_1+z-1 \end{cases}$$

$H \dot{+} z \qquad z \in [0,1)$

$$q_2 \dot{+} z \to \begin{cases} q_2+z \\ q_2+z-1 \end{cases}$$

$x \sim y$ if $x - y \in \mathbb{Q}$

Exercise $x \sim y \iff x, y$ live in the same coset.

How many cosets: uncountably many.