$m^*(A) \longrightarrow [0,\infty]$

$$\boxed{\begin{gathered} m^*(E \cup F) = m^*(E) + m^*(F) \\ \text{if } E, F \text{ are disjoint} \end{gathered}}$$

$$\overset{A}{m^*(I)} = \overset{A}{m^*(I \cap E)} + \overset{A}{m^*(I \cap E^c)}$$

$\forall$ bounded open intervals I

$b - a$ if $I = (a,b)$

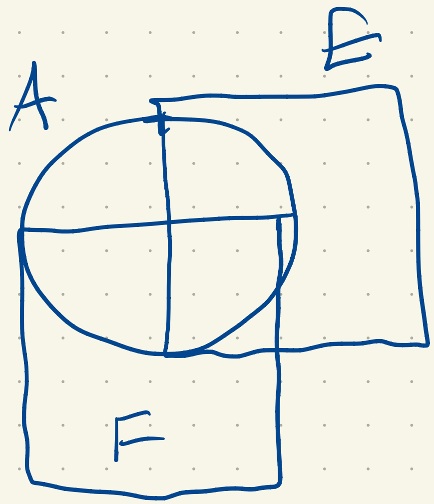
$$m^*(E \cup F) = m^*(E) + m^*(F)$$

if E, F are disjoint and measurable

algebras of sets $\longrightarrow$ closed under finite set operations

σ-algebras of sets $\longrightarrow$ " " countable " " "

If $E, F \in \mathfrak{M}$ then $E \cup F$ is measurable.



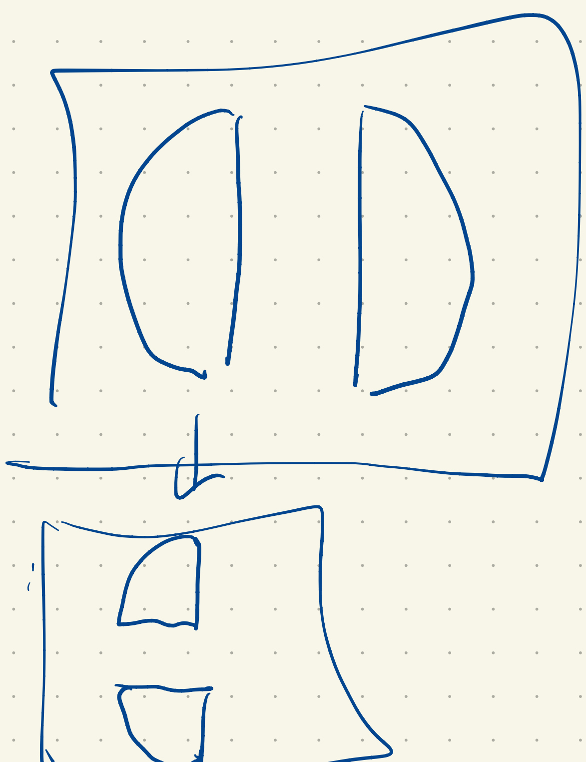
$E, F \in \mathfrak{M}$

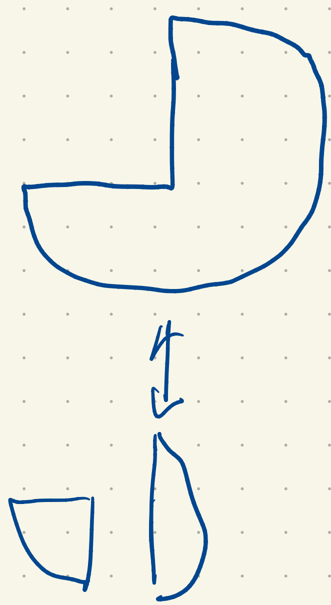
want $E \cup F$ is meas.

$$m^*(A) \geq m^*(A \cap (E \cup F)) + m^*(A \cap (E \cup F)^c)$$

$m^*(A) \leq$ ______

is free.





$A \cap (E \cup F)$

$\mathcal{M}$ is an algebra.

Lemma: Let $\{E_i\}_{i=1}^{n}$ be disjoint measurable sets

Then for all $A \subseteq \mathbb{R}$

$$m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right) = \sum_{i=1}^{n} m^*(A \cap E_i)$$

Pf: We proceed by induction on n; the case $n=1$ is obvious. Suppose the result holds for some n.

Consider $n+1$ measurable sets E_i.

Let $A \subseteq \mathbb{R}$. disjoint

Since E_{n+1} is measurable

$$m^*\left(A \cap \bigcup_{i=1}^{n+1} E_i\right) = m^*\left(A \cap \left(\bigcup_{i=1}^{n+1} E_i\right) \cap E_{n+1}\right)$$

$$+ m^*\left(A \cap \left(\bigcup_{i=1}^{n+1} E_i\right) \cap E_{n+1}^c\right)$$

$$= m^*(A \cap E_{n+1}) + m^*\left(A \cap \bigcup_{i=1}^{n} E_i\right)$$

$$= m^*(A \cap E_{n+1}) + \sum_{i=1}^{n} m^*(A \cap E_i)$$

$$= \sum_{i=1}^{n+1} m^*(A \cap E_i)$$

Prop: Suppose $\{E_i\}_{i=1}^{\infty}$ are disjoint measurable sets. Then $\bigcup_{i=1}^{\infty} E_i$ is measurable.

Pf: Let $A \subseteq \mathbb{R}$. Let $E = \bigcup_{i=1}^{\infty} E_i$. For each n

$$m^*(A) = m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right) + m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right)^c\right)$$

$$\geq m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right) + m^*\left(A \cap E^c\right) \quad \text{monotonicity}$$

$$= \sum_{i=1}^{n} m^*(A \cap E_i) + m^*(A \cap E^c)$$

This holds for all n and hence

$$m^*(A) \geq \sum_{i=1}^{\infty} m^*(A \cap E_i) + m^*(A \cap E^c)$$

and therefore

$$m^*(A) \geq m^*\left(A \cap \left(\bigcup_{i=1}^{\infty} E_i\right)\right) + m^*(A \cap E^c)$$

countable s.a.

$$= m^*(A \cap E) + m^*(A \cap E^c).$$

The reverse inequality is obvious so E is measurable.

So, what if E_i's are measurable but not disjoint?

$$F_1 = E_1$$

$$F_2 = (E_1 \cup E_2) \setminus F_1 \qquad F_1 \cup F_2 = E_1 \cup E_2$$

$$F_3 = (E_1 \cup E_2 \cup E_3) \setminus (F_1 \cup F_2) \quad F_1 \cup F_2 \cup F_3 = \bigcup_{i=1}^{3} E_i$$

Rinse + repeat: F_k $\qquad \bigcup_{k=1}^{n} F_k = \bigcup_{k=1}^{n} E_k$

disjoint

measurable if each E_i is.

$$\boxed{\bigcup_{k=1}^{\infty} F_k} = \bigcup_{k=1}^{\infty} E_k$$

$\downarrow$

measurable.

Thm: $\mathcal{M}$ is a σ-algebra.

$$m^*\big|_{\mathcal{M}} = m \rightarrow \text{Lebesgue measure.}$$

m satisfies 1) — 7)

1) — 6) / 5)

Measurable sets & Topology

$I = (a, \infty)$ is measurable.

$(b, c) \cap I$ $\qquad\qquad$ $(b, c) \cap I^c$

ϕ $\qquad\qquad$ (b, c)

(b, c) $\qquad\qquad$ ϕ

(a, c) $[b, a]$

$+$ $a - b$

$c - b$

Exercise: All intervals are measurable.

* Real every open set in $\mathbb{R}$ is a countable union of open intervals.

$\Rightarrow$ open sets are measurable.

$\Rightarrow$ closed sets are measurable

G_δ — countable intersections of open sets

F_σ → countable unions of closed sets

all measurable

Prop: TFAE

1) $E \subseteq \mathbb{R}$ is measurable

2) $\forall \varepsilon > 0$ there exists an open set $U \supseteq E$ such that $m^*(U \setminus E) < \varepsilon$.

3) $\exists$ a G_δ set $G \supseteq E$ such that $m^*(G \setminus E) = 0$.

"every measurable set is almost an open set"

Prop: TFAE

4) $\forall\, \varepsilon > 0$ $\exists$ a closed set F with $E \supseteq F$ and $m^*(E \setminus F) < \varepsilon$

5) There exists an F_σ set $F \subseteq E$ with $m^*(E \setminus F) = 0$.

6) $\forall\, \varepsilon > 0$ There exist an open set U and a closed set F with $U \supseteq E \supseteq F$ and $m^*(U \setminus F) < \varepsilon$.