This holds for all $\varepsilon > 0$ so

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m^*(E_k).$$

$$m^*(A \cup B) = m^*(A) + m^*(B)$$

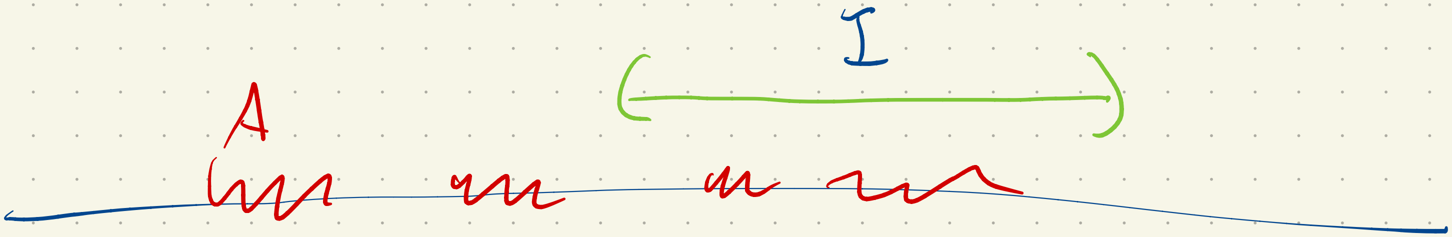
A, B disjoint

$$m^*(A \cup B) \leq m^*(A) + m^*(B)$$

$$m^*(A \cup B) < m^*(A) + m^*(B).$$

$A \subseteq \mathbb{R}$ How do we know if $m^*(A)$ is an "over estimate"

$m^*(I) = b - a$



$I \cap A \qquad I \cap A^c$

$$\underbrace{m^*(I)}_{= b-a} = m^*(I \cap A) + m^*(I \cap A^c)$$

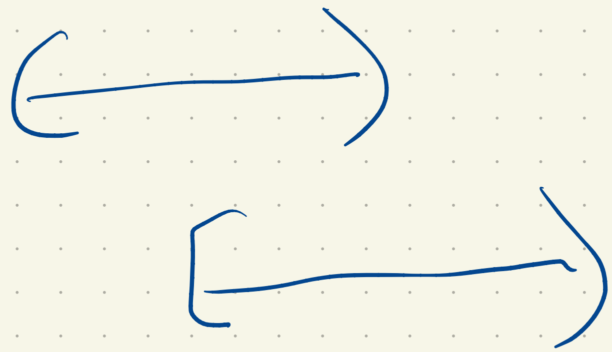
Def: A set $E \subseteq \mathbb{R}$ is <u>measurable</u> if for all bounded open intervals,

$$m^*(I) = m^*(I \cap E) + m^*(I \cap E^c)$$

We'll call the test in the above definition condition CC.

Exercise: Prove that bounded intervals are measurable.

Prove that any intervals are measurable.



$I \cap E \subseteq E$

Exercise: Null sets are measurable. $0 \leq m^*(I \cap E) \leq m^*(E) = 0$

E: null set

$$m^*(I) \overset{\leq}{=} m^*(I \cap E) + m^*(I \cap E^c)$$

$$\boxed{m^*(I) \geq \underbrace{m^*(I \cap E)}_{=0} + m^*(I \cap E^c)}$$

Prop: A set $E \subseteq \mathbb{R}$ is measurable if and only if

for all $A \subseteq \mathbb{R}$

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

CC

$\uparrow$

"Caratheodory condition"

$CC' \Leftrightarrow CC$

Pf: One direction is obvious ($CC \Rightarrow CC'$).

Suppose E is measurable. Let $A \subseteq \mathbb{R}$ and let $\varepsilon > 0$.

Pick a measuring cover $\{I_n\}$ for A such that

$$\left(\sum_{n=1}^{\infty} \ell(I_n)\right) \leq m^*(A) + \varepsilon.$$

Observe

$$\sum_{n=1}^{\infty} \ell(I_n) = \sum_{n=1}^{\infty}\left[m^*(I_n \cap E) + m^*(I_n \cap E^c)\right]$$

$$= \sum_{n=1}^{\infty} m^*(I_n \cap E) + \sum_{n=1}^{\infty} m^*(I_n \cap E^c)$$

$$\bigcup_{n=1}^{\infty}(I_n \cap E) = \left(\bigcup_{n=1}^{\infty} I_n\right) \cap E \supseteq A \cap E$$

$$\geq m^*\left(\bigcup_{n=1}^{\infty}(I_n \cap E)\right) + m^*\left(\bigcup_{n=1}^{\infty}(I_n \cap E^c)\right)$$

$$\geq m^*(A \cap E) + m^*(A \cap E^c).$$

Hence $m^*(A) + \varepsilon \geq m^*(A \cap E) + m^*(A \cap E^c)$.

This is true for all $\varepsilon > 0$ and

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$$

The reverse inequality follows from subadditivity and we obtain equality.

Def: (alt) A set $E \subseteq \mathbb{R}$ is measurable if for all $A \subseteq \mathbb{R}$

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

Prop: Suppose E_1, E_2 are measurable and disjoint. Then

$$m^*(E_1 \cup E_2) = m^*(E_1) + m^*(E_2).$$

Pf: $m^*(E_1 \cup E_2) = m^*(E_1 \cup E_2 \cap E_1) + m^*(E_1 \cup E_2 \cap E_1^c)$

$$= m^*(E_1) + m^*(E_2).$$

Exercise: If either of E or F is measurable and they are disjoint

$$m^*(E \cup F) = m^*(E) + m^*(F).$$

Notation $\mathcal{M} \subseteq \mathcal{P}(\mathbb{R})$

$\hookrightarrow$ measurable sets.

intervals, null sets.

Goal: expand this greatly via set operations.

Def: An algebra of subsets of an abstract set $\mathcal{A}$ is a collection that is closed under

- pairwise unions
- pairwise intersections
- complements

$\mathcal{F} \subseteq \mathcal{P}(\mathcal{A})$

$A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$ •

$A \cap B \in \mathcal{F}$

$A^c \in \mathcal{F}$ •

$A \cap B = (A^c \cup B^c)^c$

Exercise: Algebras are closed under finite unions/intersections

Def: A σ-algebra is an algebra that is closed under countable unions (and hence also countable intersections).

e.g. Let $\mathcal{A}$ be the collection of subsets of $\mathbb{R}$ that are either finite or have finite complement.

Algebra of finite and cofinite sets

$E_n = \{n\}$ $\qquad \bigcup_{n=1}^{\infty} E_n = \mathbb{N}$

It is not a σ-algebra.

We will show $\mathcal{M}$ is a σ-algebra.

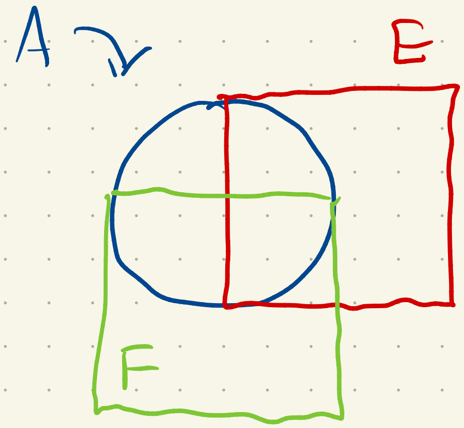
Easy $\mathcal{M}$ is closed under complements.

Let $E \in \mathcal{M}$. We want to show $E^c \in \mathcal{M}$.

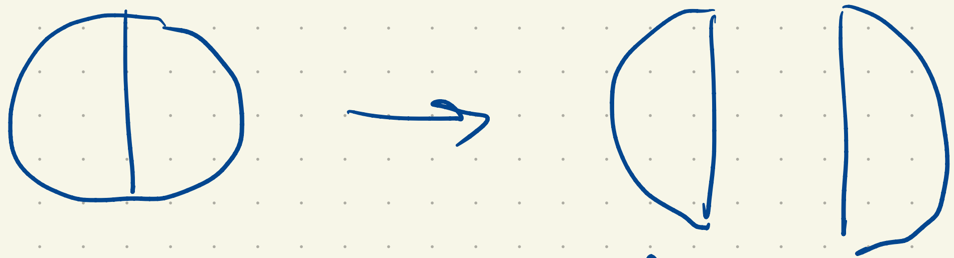
$$m^*(A) = m^*(A \cap E^c) + m^*(A \cap \underbrace{(E^c)^c}_{E})$$

✓

$E, F \in \mathcal{M}$ we want to show $E \cup F \in \mathcal{M}$

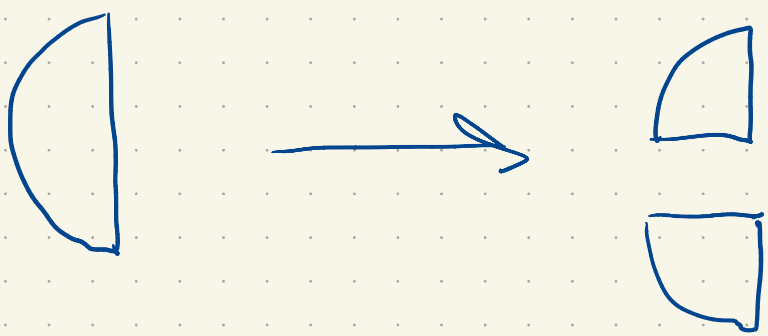


$$m^*(A) = m^*(A \cap (E \cup F)) + m^*(A \cap (E \cup F)^c)$$



$A \cap E^c$

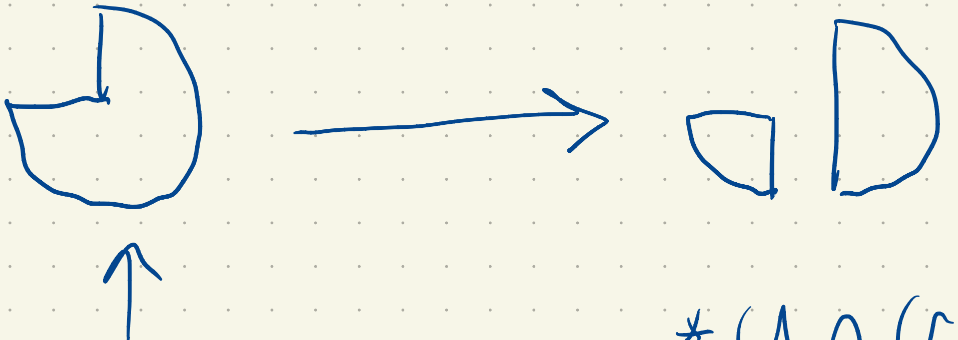
$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$



$$m^*(A \cap E^c) = m^*(A \cap E^c \cap F) + m^*(A \cap E^c \cap F^c)$$

$$E^c \cap F^c = (E \cup F)^c$$

$$m^*(A \cap E^c \cap F^c) = m^*(A \cap (E \cup F)^c)$$



$A \cap (E \cup F)$

$$m^*(A \cap (E \cup F))$$

$$= m^*(A \cap (E \cup F) \cap E) + m^*(A \cap (E \cup F) \cap E^c)$$

$$= m^*(A \cap E) + m^*(A \cap F \cap E^c)$$