finite additivity + countable s.a. $\Rightarrow$ countable additivity

Suppose $\{A_k\}$ are mutually disjoint.

Observe that for any N

$$\sum_{k=1}^{N} \ell(A_k) \underset{\text{finite add.}}{=} \ell\left(\bigcup_{k=1}^{N} A_k\right) \underset{\text{monotonicity}}{\leq} \boxed{\ell\left(\bigcup_{k=1}^{\infty} A_k\right) \underset{\text{countable s.a.}}{\leq} \sum_{k=1}^{\infty} \ell(A_k)}$$

Now take a limit as $N \to \infty$ to conclude

$s_n \to L$, $s_n \leq M \Rightarrow L \leq M$

$$\sum_{k=1}^{\infty} \ell(A_k) \leq \ell\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \ell(A_k)$$

and hence we have equality.

m^* ← outer measure.

Def: Let $E \subseteq \mathbb{R}$. A <u>measuring</u> <u>cover</u> of E is a countable collection of bounded open intervals $\{I_k\}_{k=1}^{\infty}$ such that

$$E \subseteq \bigcup_{k=1}^{\infty} I_k \qquad (\phi \text{ is an interval})$$

Def: $m^*(E) = \inf \left\{ \sum_{k=1}^{\infty} \ell(I_k) : \{I_k\}_{k=1}^{\infty} \text{ is a measuring cover of } E \right\}.$

$$I = (a,b)$$

$$\ell(I) := b-a$$

Note: $m^*(E)$ is a kind of best estimate from above for the length of E.

To what extent does m^* satisfy 1) – 7)

$m^*(I) = \ell(I)$ if I is any bounded interval.

1) $m^*([a,b]) = b-a$.

2) $m^*(E+c) = m^*(E)$ on HW, easy

3) $m^*(rE) = r\,m^*(E)$ $r>0$

I_5 $\left(6-\frac{1}{2^6}, 6+\frac{1}{2^6}\right)$

$\left(5-\frac{\varepsilon}{2^5}, 5+\frac{\varepsilon}{2^5}\right)$

5 6

If E is countable $m^*(E)$

$\ell(I_5) = \frac{2}{k}$ $\ell(I_6) = \frac{2}{k}$

$\mathbb{N}$

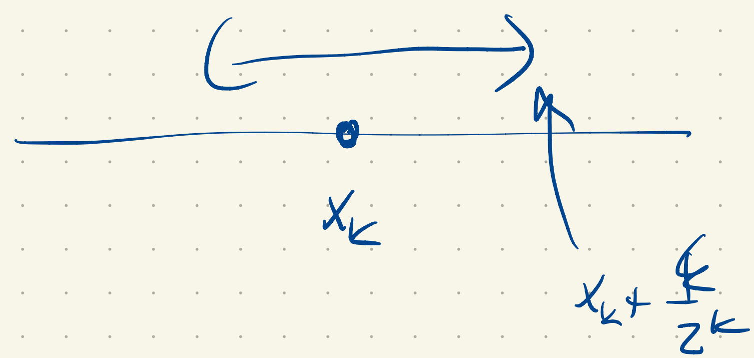
$\sum \ell(I_j) = \sum_{j=1}^{\infty} \frac{2}{k}$

$= \infty$

$$\sum_{k=1}^{\infty} \ell(I_k) = \sum_{k=1}^{\infty} \frac{2}{2^k} = 2\varepsilon$$

$$m^*(\mathbb{N}) = 0$$

$\{x_k\}_{k=1}^{\infty}$ $\qquad$ $I_k = \left(x_k - \frac{\varepsilon}{2^k}, x_k + \frac{\varepsilon}{2^k}\right)$



If E is countable $m^*(E) = 0$.

We'll later see $m^*(\Delta) = 0$.

We say E is a null set if $m^*(E) = 0$.

A property that holds for all $x \in \mathbb{R}$ except for $x \in E$ for some null set E is said to hold <u>almost everywhere</u>.

4) monotonicity.

$$E \subseteq F$$

$$m^*(E) \leq m^*(F)$$

any measuring cover for F is a measuring cover for E

Exercise: show that monotonicity holds.

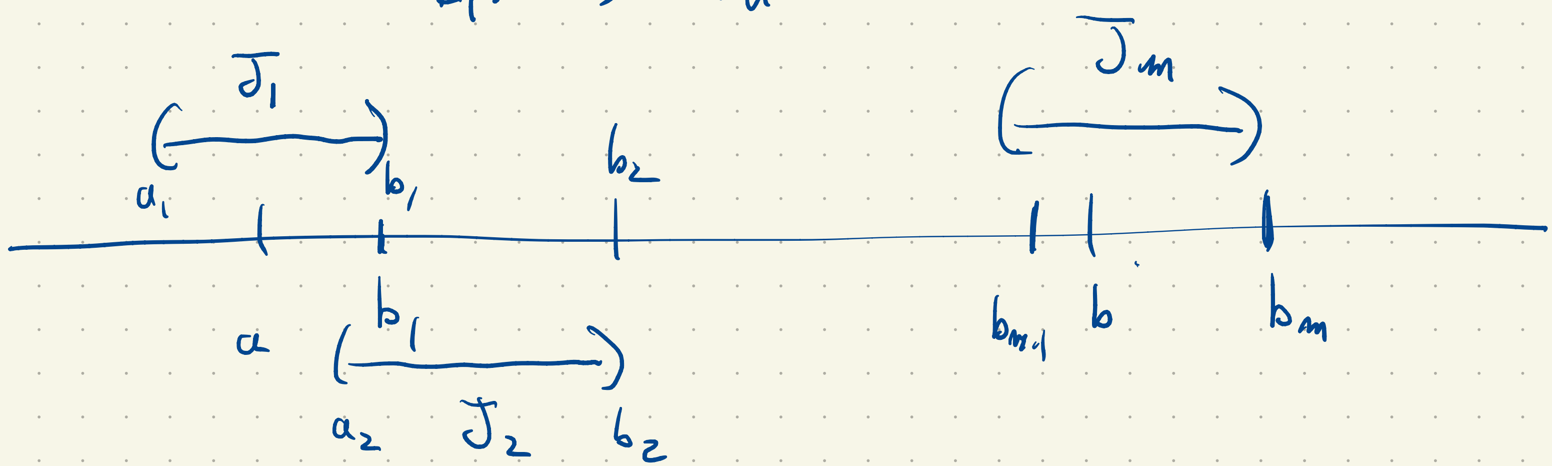
$\boxed{1)}$ 2), 3), 4) $\boxed{5}$ $\boxed{6}$ $\boxed{7}$

Theorem: If $a < b$ $m^*([a,b]) = b-a$.

Pf: Consider the measuring cover of $[a,b]$ consisting of the single set $(a-\varepsilon, b+\varepsilon)$. $\ell((a-\varepsilon, b+\varepsilon)) = b-a+2\varepsilon$ and hence $m^*([a,b]) \leq b-a+2\varepsilon$ $\forall \varepsilon > 0$.

Hence $m^*([a,b]) \leq b-a$.

Conversely, suppose $\{I_k\}$ is a measuring cover of $[a,b]$. Since $[a,b]$ is compact we can extract a finite subcover $I_{k_1}, \ldots, I_{k_n}$.



$$\ell(J_1) \geq b_1 - a \qquad (a_1 < a < b_1$$

$$\ell(J_2) \geq b_2 - b_1$$

$$\vdots$$

$$\ell(J_m) \geq b_m - b_{m-1} \geq b - b_{m-1}$$

$$(b_3 - b_2)$$

$$\sum \ell(J_k) \geq (b_1 - a) + (b_2 - b_1) + \cdots + (b - b_{m-1})$$

$$\geq b - a$$

The J_k's are distinct so

$$\sum_{k=1}^{n} \ell(I_{n_k}) \geq \sum_{j=1}^{m} \ell(J_j) \geq (b-a).$$

So $$\sum_{n=1}^{\infty} \ell(I_n) \geq \sum_{k=1}^{n} \ell(I_{n_k}) \geq (b-a).$$

Thus $m^*([a,b]) \geq b-a$.

Prop: m^* is countably subadditive.

Pf: Let E_k be a collection of sets in $\mathbb{R}$.

Let $\varepsilon > 0$. Let $\{I_{n,k}\}_{n=1}^{\infty}$ be a measuring cover for E_k such that $\sum_{n=1}^{\infty} \ell(I_{n,k}) \leq m^*(E_k) + \frac{\varepsilon}{2^k}$.

Observe $\{I_{n,k}\}_{n,k=1}^{\infty}$ is a measuring cover for $\bigcup_{k=1}^{\infty} E_k$.

Moreover $m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \ell(I_{n,k}) \leq \sum_{k=1}^{\infty}\left[m^*(E_k) + \frac{\varepsilon}{2^k}\right]$

$= \left[\sum_{k=1}^{\infty} m^*(E_k)\right] + \varepsilon$.

This holds for all $\varepsilon > 0$ so

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m^*(E_k).$$

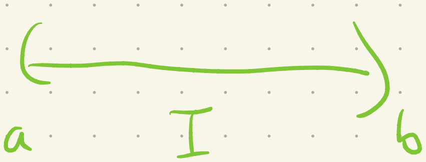
$$m^*(A \cup B) = m^*(A) + m^*(B)$$

A, B disjoint

$$m^*(A \cup B) \leq m^*(A) + m^*(B)$$

$$m^*(A \cup B) < m^*(A) + m^*(B).$$

$A \subseteq \mathbb{R}$ How do we know if $m^*(A)$ is an "over estimate"



$m^*(I) = b - a$