Suppose $f \in R[a,b]$. Then $0 \vee f$ is as well.

Let $\varepsilon > 0$. Pick step functions G, g with $g \leq f \leq G$ and $\int_a^b G - g < \varepsilon$. Observe

$$0 \vee g \leq 0 \vee f \leq 0 \vee G$$

and that $0 \vee g,\ 0 \vee G \in \mathrm{Step}[a,b]$.

Moreover,

$$\begin{aligned} 0 \vee G &= 0 \vee (G - g + g) \\ &\leq 0 \vee (G-g) + 0 \vee g \\ &= G - g + 0 \vee g. \end{aligned}$$

needs just. Easy. Exercise.

Hence $\quad 0 \vee G - 0 \vee g \leq G - g$

and $\quad \displaystyle\int_a^b (0 \vee G - 0 \vee g) \leq \int_a^b G - g < \varepsilon.$

$$\int_a^b f = \int_a^c f + \int_c^b f \qquad \text{if} \quad a < c < b$$

and $f \in R[a,b]$.

"Hard" part:

Lemma $\quad f \in R[a,b] \iff f \in R[a,c]$ and $f \in R[c,b]$

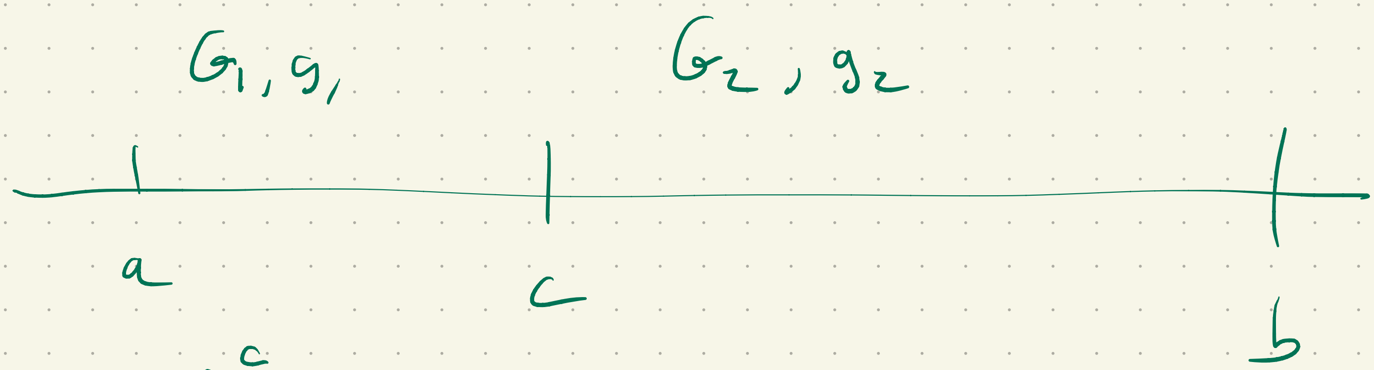
$\varepsilon > 0$

$\Rightarrow$ super easy. g, G $\quad g \le f \le G \quad \int_a^b G - g < \varepsilon.$

$$\int_a^c G - g < \varepsilon$$

$$\varepsilon > \int_a^b G - g = \int_a^c G - g + \int_c^b G - g \ge \int_a^c G - g$$

$\Leftarrow$ $\quad g_1 \le f \le G_1$



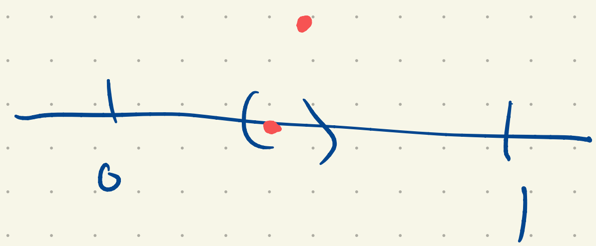
$$\int_a^c G_1 - g_1 < \varepsilon$$

$$G(x) = \begin{cases} G_1(x), & \text{if } a \le x < c \\ G_2(x), & \text{if } c \le x \le b \end{cases}$$

$$\int_a^b G = \int_a^c G + \int_c^b G = \int_a^c G_1 + \int_c^b G_2$$

There are functions that are not Riemann integrable :(

$\chi_{\mathbb{Q}}$ on $[0,1]$



HW: $\overline{\int_0^1} \chi_{\mathbb{Q}} = 1$ $\quad \underline{\int_0^1} \chi_{\mathbb{Q}} = 0$

$\int_0^1 f(x)\,dx$ $\quad [0,1]$

$$f_n \to f \Rightarrow \int_a^b f_n \to \int_a^b f$$

want

On HW: a) this works for uniform convergence.

$$f_n \in \mathcal{R}[a,b]$$

$$f_n \rightrightarrows f \Rightarrow f \in \mathcal{R}[a,b]$$

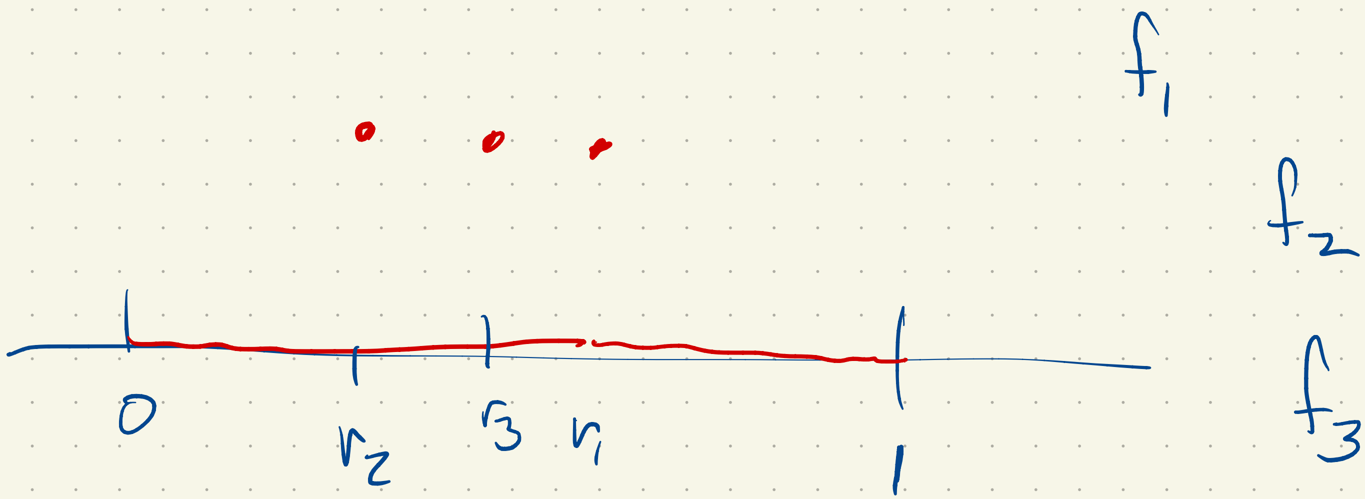
and $$\int_a^b f_n \to \int_a^b f$$

b) There are functions in $\mathcal{R}[a,b]$ that are not uniform limits of step functions.

$$\boxed{\text{The pointwise limit of riemann integrable functions need not be Riemann integrable}}$$

$$\int_a^b f_n \to \int_a^b f$$

$\chi_{\mathbb{Q}}$ on $[0,1]$ $\qquad$ $\{r_n\}$ $\qquad$ $\mathbb{Q} \cap [0,1]$



$f_k \to \chi_{\mathbb{Q}}$ $\qquad$ $\int_0^1 f_k \to \int_0^1 \chi_{\mathbb{Q}}$

Other issues: f needs to be bounded.

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$

$$\int_\varepsilon^1 \frac{1}{\sqrt{x}}\,dx \qquad \varepsilon \to 0$$

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{|x-r_k|}} \cdot \frac{1}{2^k} = f$$

$$\int_1^{\infty} \frac{1}{x^2}\,dx \qquad N$$

Lengths:

We want $\ell : P(\mathbb{R}) \longrightarrow [0, \infty]$

1) $\ell([a,b]) = b-a$

2) $\ell(A+c) = \ell(A)$

3) $\ell(rA) = r\,\ell(A)$

$r \geq 0$

geometric compatability

4) If $A \subseteq B$, $\ell(A) \leq \ell(B)$ monotonicity

5) If $A \cap B = \emptyset$ $\ell(A \cup B) = \ell(A) + \ell(B)$

Finite additivity

$A_1, \dots, A_n$ are disjoint $(A_i \cap A_j = \emptyset \quad i \neq j)$ $\ell(\cup A_i) = \sum_{i=1}^{n} \ell(A_i)$

6) Given $\{A_k\}_{k=1}^{\infty}$

$$\ell\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \ell(A_k)$$

countable subadditivity

Those are not independent

5) $\Rightarrow$ 4)

$A \subseteq B$

$B = A \cup (B \setminus A)$

disjoint

$$\ell(B) = \ell(A) + \overbrace{\ell(B \setminus A)}^{\geq 0}$$

$\geq \ell(A)$

5) + 6) $\Rightarrow$ 7)

finite additivity

countable subadditivity

countable additivity

7) If $\{A_k\}_{k=1}^{\infty}$ are mutually disjoint

Then $$\ell\left(\bigcup A_k\right) = \sum_{k=1}^{\infty} \ell(A_k)$$

Sad: You can't have 1), 2), 5) and 6) all at once.

Banach-Tarski Paradox: Let U, V be bounded open sets in $\mathbb{R}^3$.

Then there exist disjoint sets $E_1, \ldots, E_n$ and disjoint sets $F_1, \ldots, F_n$

and $U = \bigcup_{i=1}^{n} E_n$, $V = \bigcup_{i=1}^{n} F_n$ and

each E_k is congruent to F_k.

There is an isometry of $\mathbb{R}^3$ taking E_k to F_k