$f \in B[a,b]$

$\hookrightarrow$ bounded functions on $[a,b]$

$\frac{1}{\sqrt{x}}$ on $[0,1]$

If $g, G \in \text{Step}[a,b]$

$g \leq f \leq G$

We want $\int_a^b g \leq \int_a^b f \leq \int_a^b G$

$A \left\{ \int_a^b G : \begin{array}{c} G \in \text{Step}[a,b] \\ G \geq f \end{array} \right\}$

$$\overline{\int_a^b} f := \inf_{\substack{G \in \text{Step}[a,b] \\ G \geq f}} \int_a^b G$$

$\int_a^b f$

$$\underline{\int_a^b} f := \sup_{\substack{g \in Step[a,b] \\ g \le f}} \int_a^b g$$

$$B \left\{ \int_a^b g : \begin{array}{l} g \in Step[a,b] \\ g \le f \end{array} \right\}$$

$a \in A$, $b \in B$

$$\underline{\int_a^b} f \le \overline{\int_a^b} f$$

$b \le a$ $\qquad g \le f \le G$

$$\downarrow \qquad \downarrow$$

$$\int_0^1 g \le \int_0^1 G$$

If $f \in Step[a,b]$ then

$$\overline{\int_a^b} f \le \int_a^b f \le \underline{\int_a^b} f \quad \text{and}$$

$$\boxed{\overline{\int_a^b} f = \underline{\int_a^b} f.}$$

Def: $R[a,b] \subseteq B[a,b]$ is the subset of functions f such that $\overline{\int_a^b} f = \underline{\int_a^b} f$ in which case we define $\int_a^b f$ to be the common value and we say f is Riemann integrable.

$$\int_a^b f$$

Prop: Suppose $f \in B[a,b]$. Then $f \in R[a,b]$ iff $\forall \varepsilon > 0$ there exist $g, G \in \text{Step}[a,b]$ with $g \leq f \leq G$ and

$$\int_a^b G \leq \int_a^b g + \varepsilon. \qquad \left| \int_a^b G - \int_a^b g \right| \leq \varepsilon$$

$$0 \leq \int_a^b G - \int_a^b g \leq \varepsilon$$

Pf: Suppose $f \in R[a,b]$. Let $\varepsilon > 0$ and pick $g, G \in \mathrm{Step}[a,b]$ with

$$\int_a^b G \leq \left(\overline{\int_a^b} f\right) + \frac{\varepsilon}{2} \quad \text{and} \quad \int_a^b g \geq \left(\underline{\int_a^b} f\right) - \frac{\varepsilon}{2}.$$

Since $\overline{\int_a^b} f = \underline{\int_a^b} f$ we conclude $\int_a^b G \leq \int_a^b g + \varepsilon$.

Conversely, suppose that for all $\varepsilon > 0$ there exist $g, G \in \mathrm{Step}[a,b]$ with $g \leq f \leq G$ and $\int_a^b G \leq \int_a^b g + \varepsilon$. Let $\varepsilon > 0$ and pick such step functions g, G.

Observe
$$\overline{\int_a^b} f \leq \int_a^b G \leq \int_a^b g + \varepsilon \leq \underline{\int_a^b} f + \varepsilon.$$

This holds for all $\varepsilon > 0$ and thus $\overline{\int_a^b} f \leq \underline{\int_a^b} f$.

The reverse inequality is obvious so $f \in R[a,b]$.

Cor: $C[a,b] \subseteq R[a,b]$

Pf: Let $f \in C[a,b]$. Let $\varepsilon > 0$. Pick $\delta > 0$ such that

if $x, y \in [a,b]$ and $|x-y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.

Note that this uses the uniform continuity of f and is based

on the compactness of $[a,b]$.

Let P be a partition $\{x_0, \dots, x_n\}$ such that $dx_k = x_k - x_{k-1} < \delta$

for each $k = 1, \dots, n$.

Define
$$G_k = \sup_{x \in I_k} f(x)$$

$$g_k = \inf_{x \in I_k} f(x) \quad \text{where} \quad I_k = (x_{k-1}, x_k).$$

Clearly $G_k \geq g_k$. Moreover $G_k - g_k \leq \varepsilon$

since $|f(x) - f(y)| < \varepsilon$ for all $x, y \in I_k$.

Define G to be the step function that equals G_k on each I_k and equals f on P. Define g similarly.

Then $g \le f \le G$.

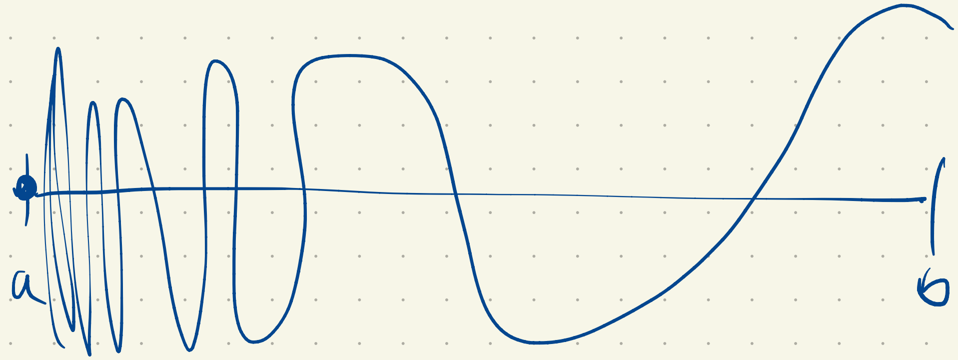
Moreover $$\int_a^b (G-g) \le \int_a^b \varepsilon = \varepsilon \cdot (b-a).$$

So $$\int_a^b G \le \int_a^b g + \varepsilon(b-a).$$

Since $\varepsilon > 0$ is arbitrary, we are done.

Exercise: Suppose $f \in B[a,b]$ and $f \in R[c,d]$ whenever
$$a < c < d < b.$$

Then $f \in R[a,b]$.



Exercise: If $f \in \mathcal{R}[a,b]$ then $f \in \mathcal{R}[c,d]$ if

$$a \le c < d \le b.$$

Properties.

2) Monotonicity. Suppose $f_1, f_2 \in \mathcal{R}[a,b]$ and $f_1 \le f_2$.

If $g \in \text{Step}[a,b]$ and $g \le f_1$ then $g \le f_2$ also

and hence $\underline{\int_a^b} f_1 \le \underline{\int_a^b} f_2$.

But then $\int_a^b f_1 \le \int_a^b f_2$.

Linearity: Suppose $f, g \in \mathcal{R}[a,b]$. Then $f+g \in \mathcal{R}[a,b]$ and

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

For all $f \in \mathcal{R}[a,b]$ then $|f| \in \mathcal{R}[a,b]$ and

$$\left| \int_a^b f \right| \leq \int_a^b |f|$$

If $f, g \in \mathcal{B}[a,b]$ $\quad (f \vee g)(x) = \max(f(x), g(x))$

Claim: $0 \vee f \in \mathcal{R}[a,b]$ if $f \in \mathcal{R}[a,b]$

$$|f| = (0 \vee f) - (0 \vee (-f))$$