for all $f \in \mathcal{F}$.

Exercise: pointwise bounded + equicts $\Rightarrow$ uniformly bounded.

Thm: Let X be compact. If $\mathcal{F} \subseteq C(X)$ is pointwise bounded and equicontinuous it is totally bounded.

Thm (Arzela-Ascoli)
Let X be compact. A subset $\mathcal{F} \subset C(X)$ is compact if and only if it is closed, pointwise bounded and equicontinuous.

Pf: Suppose $\mathcal{F} \subseteq C(X)$ is pointwise bounded and equicontinuous.

Let $\varepsilon > 0$. Pick δ so if $d(x,z) < \delta$, $|f(x) - f(z)| < \frac{\varepsilon}{4}$ for all $f \in \mathcal{F}$. Let $x_1, \dots, x_K$ be a δ-net for X, which exists since X is compact and hence totally bounded.

Pick M so that $|f(x_k)| \leq M$ for all $f \in \mathcal{F}$ and all $1 \leq k \leq K$. Let $y_1, \dots, y_J$ be an $\frac{\varepsilon}{4}$ net for $[-M, M]$.

Let P be the set of functions from $\{x_1, \dots, x_K\}$ to $\{y_1, \dots, y_J\}$. There are J^K such functions.

Given $p \in P$ let $\mathcal{F}_p = \{ f \in \mathcal{F} : |f(x_k) - p(x_k)| < \frac{\varepsilon}{4} : 1 \leq k \leq K \}$.

Observe $\bigcup_{p \in P} \mathcal{F}_p = \mathcal{F}$. $\Big(f \in \mathcal{F} \quad f(x_k) \in [-M, M]$

$|y_{j_k} - f(x_k)| < \varepsilon/4 \Big)$

Pick $f, g \in \mathcal{F}_p$. Let $x \in X$. Pick x_k so that $d(x, x_k) < \delta$.

Then

$$|f(x) - g(x)| \leq |f(x) - f(x_k)| + |\overset{p(x_k)}{f(x_k) - g(x_k)}| + |g(x_k) - g(x)|$$

$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon$$

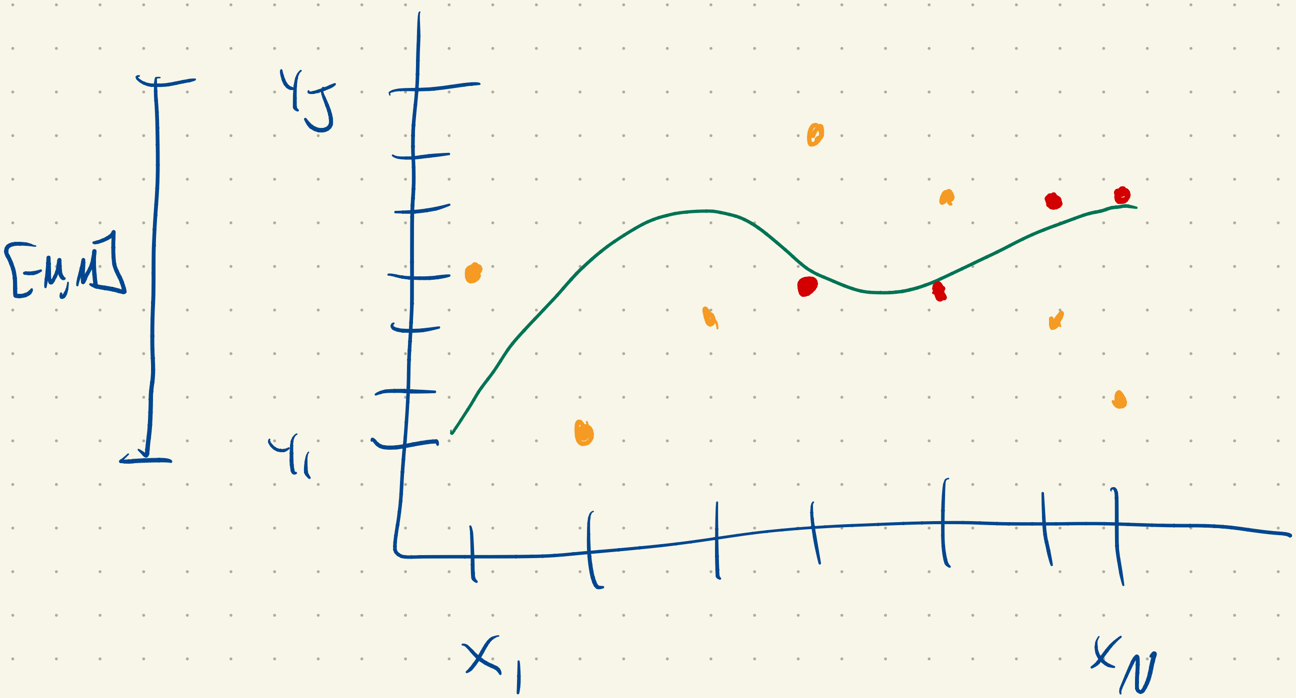
since

$$|f(x_k) - g(x_k)| \leq |f(x_k) - p(x_k)| + |p(x_k) - g(x_k)|$$

$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4}.$$

Hence $d(f, g) \leq \varepsilon$ and $\operatorname{diam}(\mathcal{F}_p) \leq \varepsilon$ as well.

Thus $\mathcal{F}$ is totally bounded.



$f \in C(X)$

$p \in P$

Integration

Riemann Integral

$[a,b] \subseteq \mathbb{R} \qquad a < b$

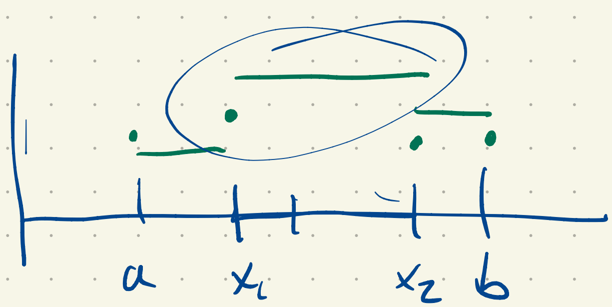
Partition: $a = \underbrace{x_0 < x_1 < \cdots < x_n}_{} = b$

$\downarrow$

$P \to$ finite subset of $[a,b]$ that contains the endpoints.

Step functions: $\text{Step}[a,b]$

$g \in \text{Step}[a,b]$ if there exists a partition P of $[a,b]$ such that g is constant on each interval (x_{k-1}, x_k).



We call such a partition a <u>step partition</u> for g.

$\int_a^b g$ if $g \in \text{Step}[a,b]$.

1) pick a step partition for g

$$P: a = x_0 < x_1 \cdots < x_n = b$$

2) Let $dx_k = x_k - x_{k-1} \quad 1 \le k \le n$

3) $\displaystyle \overset{P}{\int_a^b} g = \sum_{k=1}^{n} g_k \, dx_k$ where g_k is the constant value of g on (x_{k-1}, x_k).

This is independent of the choice of step partition. How?

We say $\mathcal{P}'$ is a refinement of $\mathcal{P}$ if $\mathcal{P}' \supseteq \mathcal{P}$.

If $\mathcal{P}_1$ and $\mathcal{P}_2$ are partitions we call $\mathcal{P}_1 \cup \mathcal{P}_2$ the common refinement of the two.

$$\overset{\mathcal{P}_1}{\int_a^b g} = \overset{\mathcal{P}_2}{\int_a^b g} \qquad \overset{\mathcal{P}_1 \cup \mathcal{P}_2}{\int_a^b g}$$

If $\mathcal{P}'$ is a refinement of $\mathcal{P}$ $\overset{\mathcal{P}'}{\int_a^b g} = \overset{\mathcal{P}}{\int_a^b g}$.

Exercise: proof by induction on the size of

$\mathcal{P}' \setminus \mathcal{P}$.

Properties:

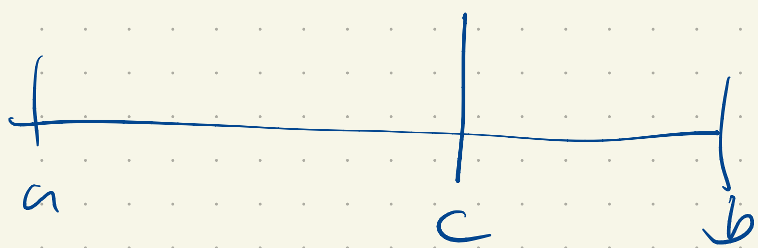
* Exercise 1) Linearity

2) Monotonicity $g_1, g_2 \in Step[a,b]$ $g_1 \le g_2 \Rightarrow \int_a^b g_1 \le \int_a^b g_2$

3) $\left| \int_a^b g \right| \le \int_a^b |g|$

4) If $c \in (a,b)$ then

$$\int_a^b g = \int_a^c g + \int_c^b g \qquad \Big(\underbrace{g \in Step[a,b]}_{g|_{[a,c]} \in Step[a,c]}\Big)$$



Monotonicity. Suppose $g, \hat{g} \in \text{Step}[a,b]$ and $g \le \hat{g}$

Let P be a step partition for both g and $\hat{g}$.

Then $$\int_a^b g = \sum_{k=1}^{n} g_k \, dx_k \le \sum_{k=1}^{n} \hat{g}_k \, dx_k = \int_a^b \hat{g}$$

$$g \in \text{Step}[a,b] \Rightarrow |g| \in \text{Step}[a,b]$$

$$-|g| \le g \le |g|$$

$$\int_a^b -|g| \le \int_a^b g \le \int_a^b |g|$$

$$\begin{aligned} &\parallel \\ -&\int_a^b |g| \end{aligned}$$

$$\left| \int_a^b g \right| \le \int_a^b |g|$$