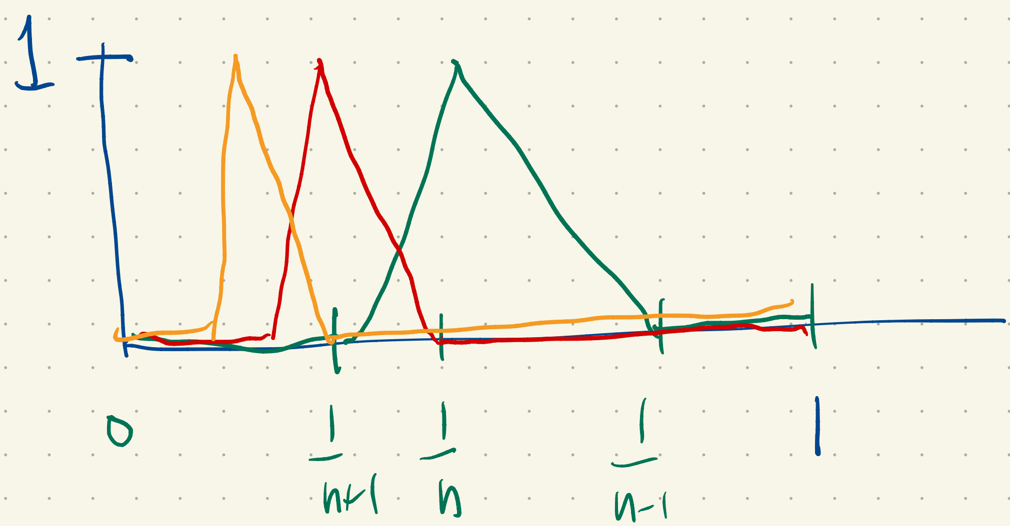
$C[0,1] \longrightarrow$ compact subsets

$C(X)$ $\quad X$ is compact

$\underbrace{\text{complete}}_{\Downarrow \atop \text{closed}}$, $\underbrace{\text{totally bounded subsets}}$

bounded $\not\Rightarrow$ totally bounded (no Cauchy subsequence!)



$f_n \quad n \geq 2$

$$d(f_n, f_m) = 1 \quad n \neq m$$

Def: A subset $\mathcal{F} \subseteq C(X)$ where X is a metric space is equicontinuous if for every $\varepsilon > 0$ there exists $\delta > 0$ so that if $x, y \in X$ with $d(x,y) < \delta$ then $|f(x) - f(y)| < \varepsilon$ for all $f \in \mathcal{F}$.

Each f in an equicontinuous family is uniformly continuous.

Equicontinuity: one δ works everywhere for all members of the family.

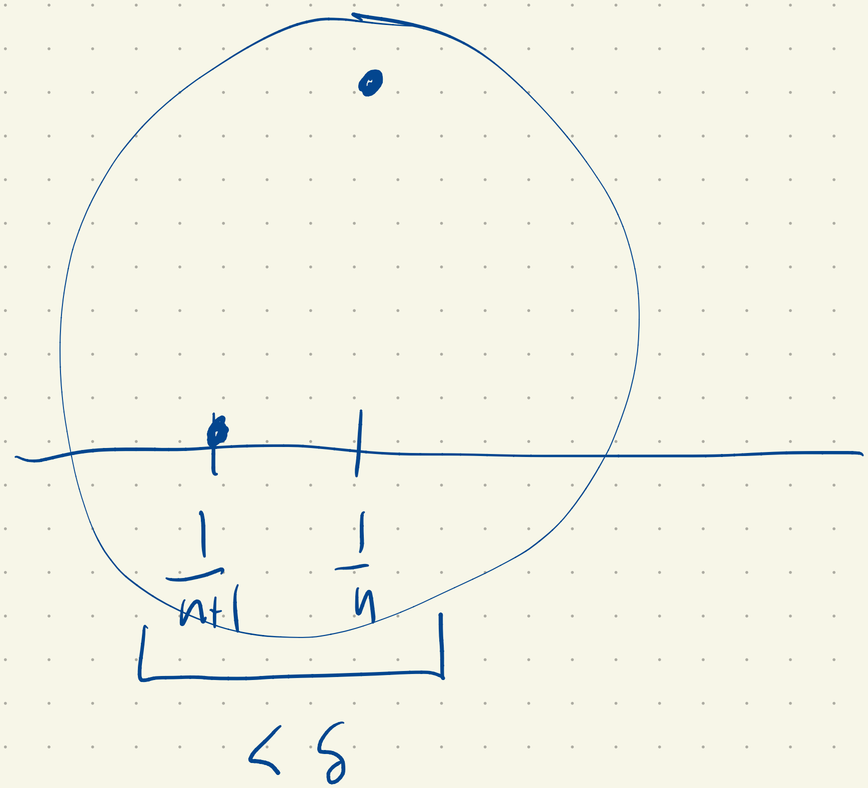
Not equicontinuous: $\exists\, \varepsilon_0 > 0$ so that for all $\delta > 0$

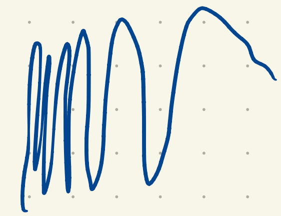
there exists $x, y \in X$ and $f \in \mathcal{F}$ such that

$d(x,y) < \delta$ but $|f(x) - f(y)| \geq \varepsilon_0$.

Bad $\varepsilon_0 = 1$

$\delta > 0$





$[0,1]$ $\sin(1/x)$

Lip $[a,b]$ lip functions

$\mathrm{Lip}_K[a,b]$

$$|f(x)-f(y)| \le K|x-y|$$

$\varepsilon > 0$

Pick δ so $K\delta < \varepsilon$

Then if $f \in \mathrm{Lip}_K[a,b]$

and if $x,y \in [a,b]$ with $|x-y| < \delta$

then
$$\begin{aligned} |f(x)-f(y)| &\le K|x-y| \\ &< K\delta \\ &< \varepsilon \end{aligned}$$

Boundedness and equicontinuity are independent.

not bounded, not equicts	:	$C[0,1]$
bounded, not equicts	:	$B_1(0)$
not bounded, equicts	:	$Lip_K[0,1]$
bounded, equicts	:	$\overline{B}_1(0) \cap Lip_K[0,1]$

Equicontinuity is necessary for a family of functions to be totally bounded.

Thm: Let X be a compact metric space. Every totally bounded subset $\mathcal{F}$ of $C(X)$ is equicontinuous.

Pf: Let $\varepsilon > 0$. Let $f_1, \dots, f_n$ be an $\frac{\varepsilon}{3}$-net for $\mathcal{F}$. Each f_k in the net is uniformly continuous, so there exists δ_k so that if $x, y \in X$ and $d(x,y) < \delta_k$,

$$|f_k(x) - f_k(y)| < \frac{\varepsilon}{3}.$$

Let $\delta = \min \delta_k$, so $\delta > 0$. Suppose $x, y \in X$ and $d(x,y) < \delta$. Let $f \in \mathcal{F}$. Then there exists f_k in the net so that $\|f - f_k\|_\infty < \frac{\varepsilon}{3}$. Then

$$|f(x) - f(y)| \leq |f(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f_k(y) - f(y)|$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$

$= \varepsilon.$ $\square$

We will see that $\underbrace{\text{boundedness}}_{\substack{\downarrow \\ \ell_\infty \text{ (uniform boundedness)}}}$ + equicontinuity $\Rightarrow$ total boundedness

if X is compact

We will prove in fact pointwise boundedness + equicontinuity $\Rightarrow$ total boundedness.

$\mathcal{F} \subseteq C(X)$ $\quad$ $\forall x \in X$ there exists M_x

such that $|f(x)| \leq M_x$

for all $f \in \mathcal{F}$.

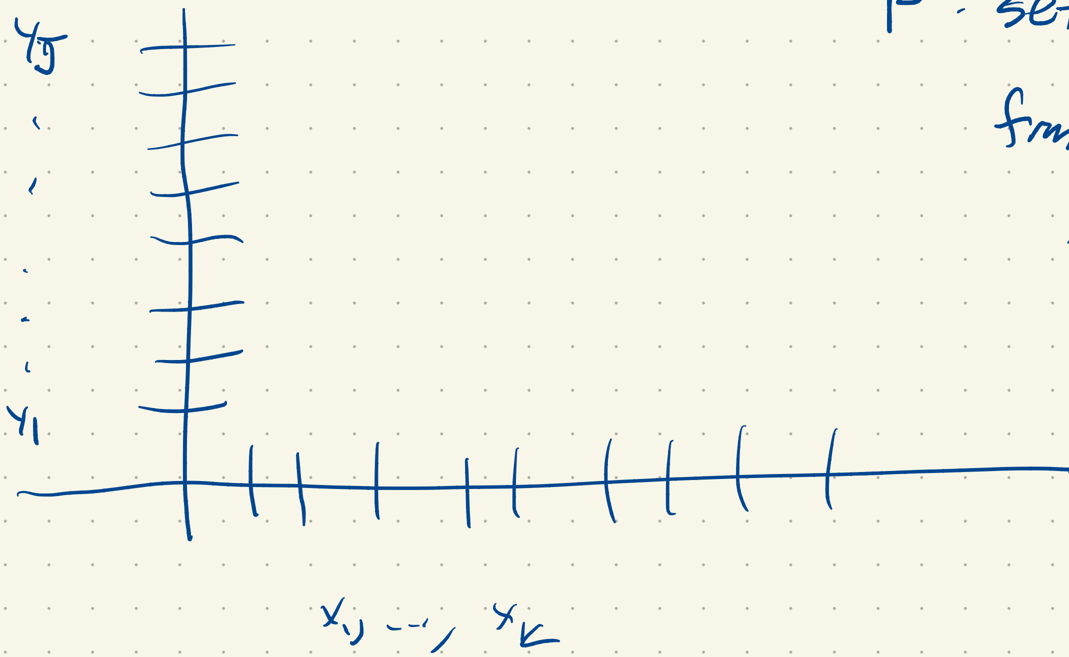
Exercise: pointwise bounded + equicts $\Rightarrow$

uniformly bounded.

Thm: Let X be compact. If $\mathcal{F} \subseteq C(X)$ is pointwise bounded and equicontinuous it is totally bounded.

Thm (Arzela-Ascoli)

Let X be compact. A subset $\mathcal{F} \subset C(X)$ is compact if and only if it is closed, pointwise bounded and equicontinuous.



P: set of functions from $\{x_1, \dots, x_K\}$ to $\{y_1, \dots, y_J\}$

$|P| = J^K$

finite

Given $p \in P$

$$\mathcal{F}_p = \left\{ f \in \mathcal{F} : \left| f(x_k) - p(x_k) \right| < \frac{\varepsilon}{4}, \ 1 \le k \le K \right\}$$

Goal: $\operatorname{diam}(\mathcal{F}_p) < \varepsilon$.

$$\mathcal{F} \subseteq \bigcup_{p \in P} \mathcal{F}_p$$