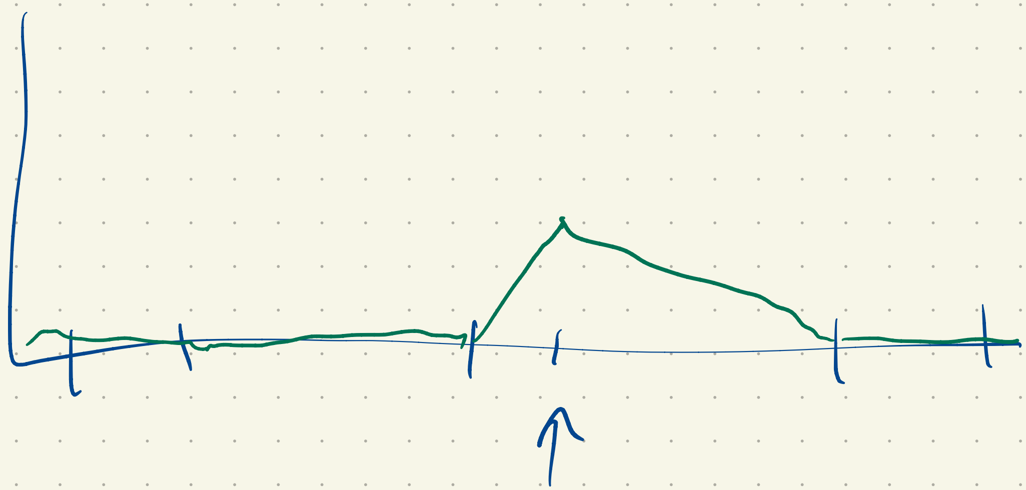
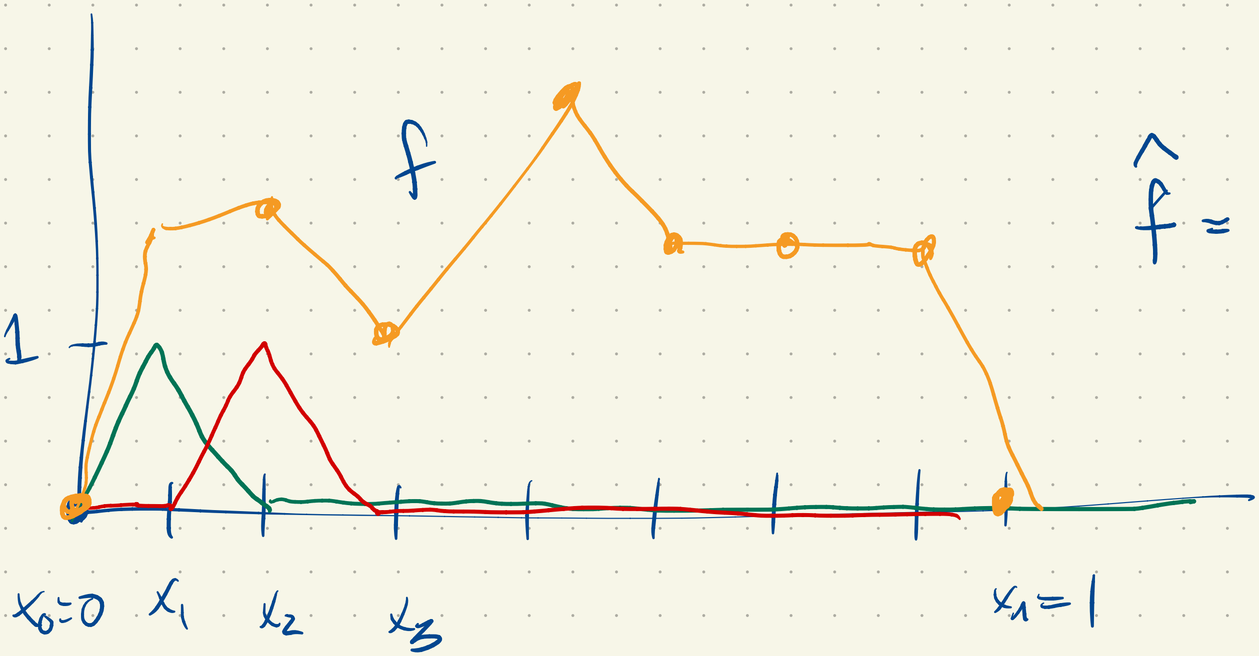




$$\hat{f} = \sum_{k=1}^{n-1} a_k H_k$$

$$= \sum_{k=1}^{n-1} f(x_k) H_k$$

$$H_k(x_\ell) = \begin{cases} 1 & k=\ell \\ 0 & k\neq\ell \end{cases}$$

$$H_1, H_2, \ldots, H_{n-1} \in \overline{P[0,1]}$$

$$\hat{f}(x_\ell) = \sum_{k=1}^{n-1} f(x_k) H_k(x_\ell) = f(x_\ell)$$

Lemma: There exists a sequence $P_k(x)$ of polynomials on $[0,1]$ converges uniformly to $\sqrt{x}$.

Cor: $\text{abs} \in \overline{P[0,1]}$.

Pf: Let $\varepsilon > 0$. Let P_n be a polynomial such that

$$\left|\sqrt{x} - P_n(x)\right| < \varepsilon \text{ for all } x \in [0,1].$$

Then if $z \in [-1,1]$, $z^2 \in [0,1]$ and

$$\left|\sqrt{z^2} - P_n(z^2)\right| < \varepsilon \quad \text{for all } z \in [-1,1].$$

That is, $\left|\,|z| - P_n(z^2)\right| < \varepsilon$ for all $z \in [-1,1]$.

Since $P_n(z^2)$ is a polynomial in z, we are done. $\square$

Pf of lemma: For $0 \le x \le 1$ define $P_0(x) = 0$ and for $k \geq 0$, define

$$P_{k+1}(x) = P_k(x) + \frac{x - P_k^2(x)}{2}.$$

We claim that for all k $0 \le P_k(x) \le \sqrt{x}$ and that $P_{k+1}(x) \geq P_k(x)$. This is obvious when $k=0$.

Suppose this holds for some k. Then

$$P_{k+1}(x) = P_k(x) + \frac{x - P_k(x)^2}{2} \geq P_k(x) \geq 0.$$

Moreover: $P_{k+1}(x) = P_k(x) + \frac{\sqrt{x} + P_k(x)}{2} \cdot \left(\sqrt{x} - P_k(x)\right)$

$$\leq P_k(x) + 1 \cdot \left(\sqrt{x} - P_k(x)\right)$$

$$= \sqrt{x}.$$

The proof that $P_{(k+1)+1} \geq P_{k+1}$ is now the same as the above.

The sequence P_k is bounded above and pointwise monotone increasing and therefore converges pointwise to a limit P. Moreover

$$P(x) = \lim_{k \to \infty} P_{k+1}(x) = \lim_{k \to \infty} P_k(x) + \frac{x - (P_k(x))^2}{2}$$

$$= P(x) + \frac{x - (P(x))^2}{2}.$$

So for each $x \in [0,1]$ $P(x)^2 = x$ and

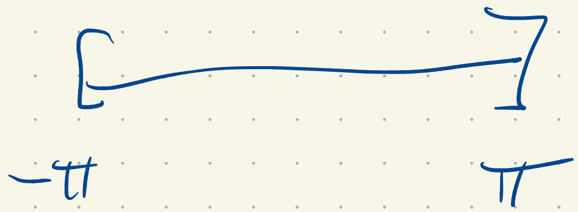
since $P(x) \geq 0$, $P(x) = \sqrt{x}$.

Since $[0,1]$ is compact and since $\sqrt{\cdot}$ is continuous,
Dini's theorem implies that the convergence is uniform.

$\square$

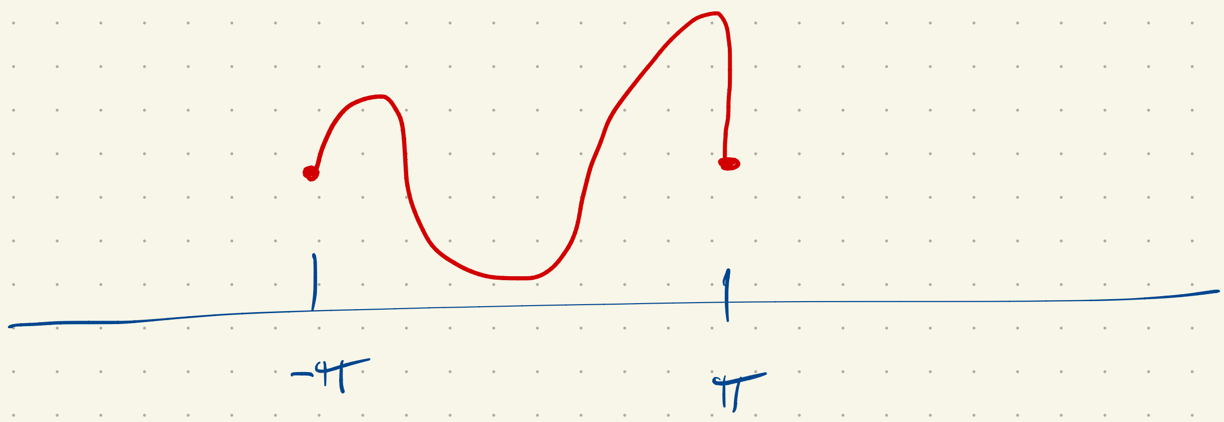
Trigonometric Polynomials

$$T(x) = a_0 + \sum_{k=1}^{n} a_k \cos(kx) + \sum_{k=1}^{n} b_k \sin(kx)$$



$f \in C[-\pi, \pi]$

$f(-\pi) = f(\pi)$



$C^{2\pi} \to$ continuous 2π-periodic functions on $\mathbb{R}$

Given $f \in C^{2\pi}$ and $\varepsilon > 0$ there exists a trig polynomial T such that $|f(x) - T(x)| < \varepsilon$ for all $x \in \mathbb{R}$.

1) The product of trig polynomials is a trig polynomial.

$$\sin(kx)\sin(mx) = \frac{1}{2}\left[\cos((k-m)x) - \cos((k+m)x)\right]$$

2) If T is a trig polynomial then $T(x-\frac{\pi}{2})$ is as well.

$$\sin\left(k\left(x-\frac{\pi}{2}\right)\right) = \sin\left(kx - \frac{k\pi}{2}\right)$$

$$= \begin{cases} \sin(kx) & k \equiv 0 \mod 4 \\ \cos(kx) & k \equiv 1 \mod 4 \\ -\sin(kx) & k \equiv 2 \mod 4 \\ -\cos(kx) & k \equiv 3 \mod 4 \end{cases}$$

Lemma: Suppose $f \in C^{2\pi}$ is even. Then for all $\epsilon > 0$ there exists a trig polynomial T such that $\|f-T\|_\infty < \epsilon$.

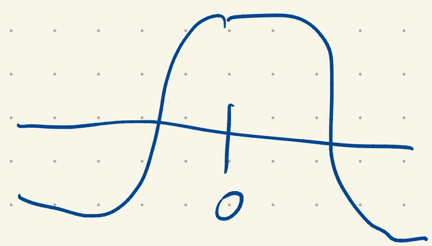
Pf: Consider $f \circ \arccos : [-1,1] \to \mathbb{R}$. This is a continuous function and thus there exists a polynomial p such that

$$\left|(f \circ \arccos)(y) - p(y)\right| < \varepsilon \quad \text{for all } y \in [-1,1].$$

But then

$$\left|f(\arccos(\cos(x))) - p(\cos(x))\right| < \varepsilon$$

for all $x \in [-\pi, \pi]$. Note that



$$\arccos(\cos(x)) = \begin{cases} x & x \in [0,\pi] \\ -x & x \in [-\pi, 0] \end{cases}$$

$$= |x|.$$

So $$\left|f(|x|) - p(\cos(x))\right| < \varepsilon \quad \text{for all } x \in [-\pi, \pi].$$

Since f is even and 2π-periodic,

$$\left| f(x) - p(\cos(x)) \right| < \varepsilon \quad \text{for all } x \in \mathbb{R}.$$

Note that $p(\cos(x))$ is a trig polynomial. $\square$

See text.