$C[a,b]$, the metric space

1) dense, useful subsets

2) how to identify compact subsets

Goal: $\overline{P[0,1]} = C[0,1]$ (closure in L_∞ sense)

a) $\overline{PL[0,1]} = C[0,1]$

$\hookrightarrow$ piecewise linear and continuous.

$0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$ and

g is linear on each $[x_{k-1}, x_k]$ $k = 1, \ldots, n$.

b) $\overline{P[0,1]} \supseteq PL[0,1]$

a) + b) $\Rightarrow$ $\overline{\overline{P[0,1]}} \supseteq \overline{PL[0,1]} = C[0,1]$

$$\overline{\overline{P[0,1]}} = \overline{P[0,1]}$$

Key tool: If $W \subseteq X$ is a subspace then $\overline{W}$ is a subspace.

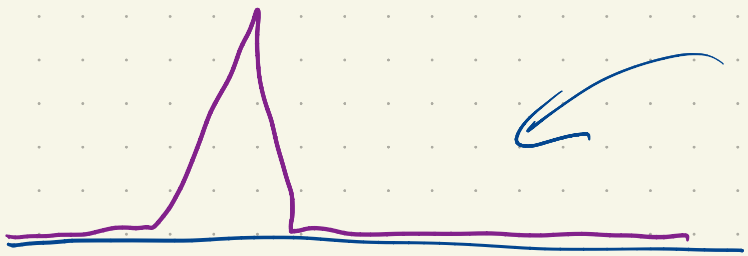
$Z \subseteq \ell_1$

$\overline{Z} = \ell_1$

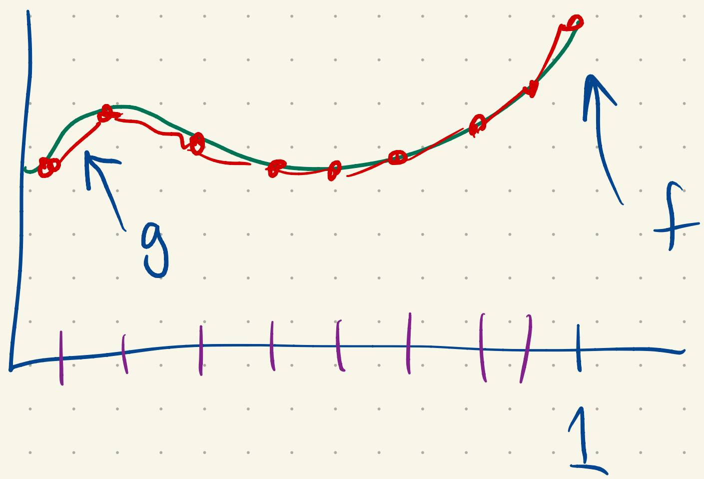
$Z \subseteq \ell_\infty$ $(1,1,1,1,1,\ldots)$

$\overline{Z} = c_0$ $0,\ldots 0)$ $\uparrow$

$$\overline{P[0,1]} = PL[0,1]$$



$\overline{P[0,1]}$



"itty bitty" ε

$$\|f-g\|_\infty \le \varepsilon$$

$$\Downarrow$$

$$|f(x)-g(x)| \le \varepsilon$$

$$\forall x \in [0,1]$$

Prop: $\overline{PL[0,1]} = C[0,1]$.

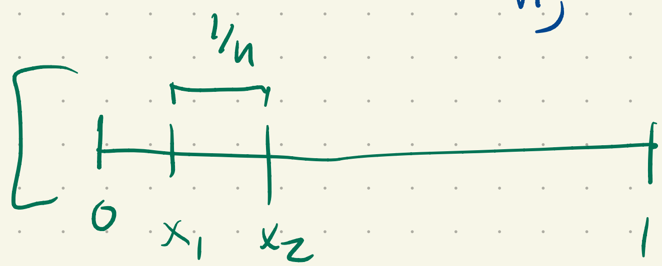
[Job: find $g \in PL[0,1]$, $\|f-g\|_\infty < \varepsilon$]

Pf: Let $f \in C[0,1]$. Let $\varepsilon > 0$.

Since $[0,1]$ is compact, f is uniformly continuous.

Hence we can pick n so that if $|x-y| \leq \frac{1}{n}$, $|f(x)-f(y)| < \varepsilon$.

Let $x_k = \frac{k}{n}$ $\quad 0 \leq k \leq n$.



Let g be the piecewise linear function such that $g(x_k) = f(x_k)$ for each k and g is linear on each $[x_{k-1}, x_k]$.

So, on such an interval

$$g(x) = f(x_{k-1}) + \left(\frac{x - x_{k-1}}{x_k - x_{k-1}}\right)\left[f(x_k) - f(x_{k-1})\right]$$

Given $x \in [x_{k-1}, x_k]$

$$
\begin{aligned}
|f(x) - g(x)| &= |f(x) - f(x_{k-1}) + f(x_{k-1}) - g(x)| \\
&\leq |f(x) - f(x_{k-1})| + |g(x) - f(x_{k-1})| \\
&= |f(x) - f(x_{k-1})| + \left|\frac{x - x_{k-1}}{x_k - x_{k-1}}\right| \, |f(x_k) - f(x_{k-1})| \\
&\leq \varepsilon + 1 \cdot \varepsilon \\
&= 2\varepsilon.
\end{aligned}
$$

$\square$

In fact, we showed that piecewise linear functions with rational partitions are dense.

Exercise: Show that PL functions with rational partitions taking on rational values at the partition endpoints are dense. Conclude $C[0,1]$ is separable.

Lemma: Let X be a normed vector space and let $W \subseteq X$ be a subspace. Then $\overline{W}$ is a subspace.

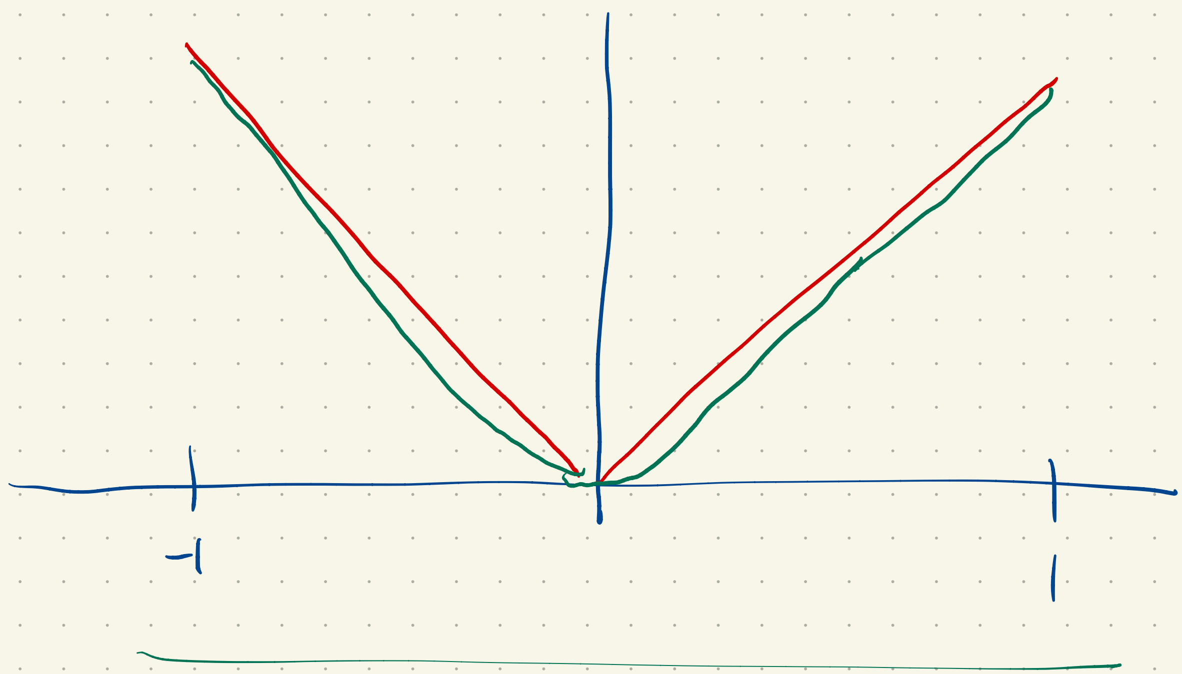
Pf: Let $x, y \in \overline{W}$. There exist (x_n) and (y_n) in W such that $x_n \to x$ and $y_n \to y$. Then

$x_n + y_n \longrightarrow x + y$. Since each $x_n + y_n \in W$, $x + y \in \overline{W}$. The argument for scalar multiplication is similar.

Prop: Given $\varepsilon > 0$ there exists a polynomial p such that

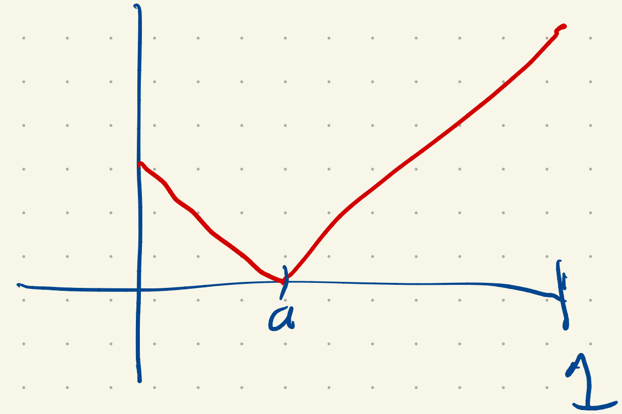
$$|\operatorname{abs}(x) - p(x)| < \varepsilon \quad \text{for all } x \in [-1, 1].$$

$$\left(\operatorname{abs} \in \overline{P[-1,1]} \right).$$



$\mathrm{abs}_a(x) = |x-a|$ on $[0,1]$

$a \in [0,1]$



Given $\varepsilon > 0$

There is a polynomial p with

$$|\mathrm{abs}_a(x) - p(x)| < \varepsilon \quad \text{for all } x \in [0,1]$$

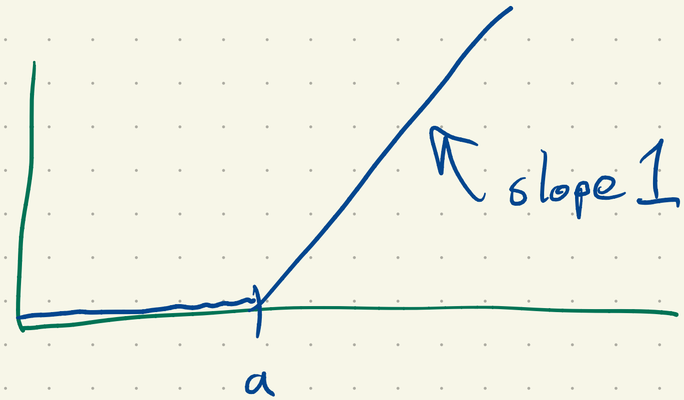
q

$$|\mathrm{abs}(x) - q(x)| < \varepsilon \quad \text{for all } x \in [-1,1]$$

$$\Big|\mathrm{abs}(\underbrace{x-a}_{\substack{\in[-1,1] \text{ if } x\in[0,1] \\ a\in[0,1]}}) - \overbrace{q(x-a)}^{p(x)}\Big| < \varepsilon \quad \text{for all } x \in [0,1]$$

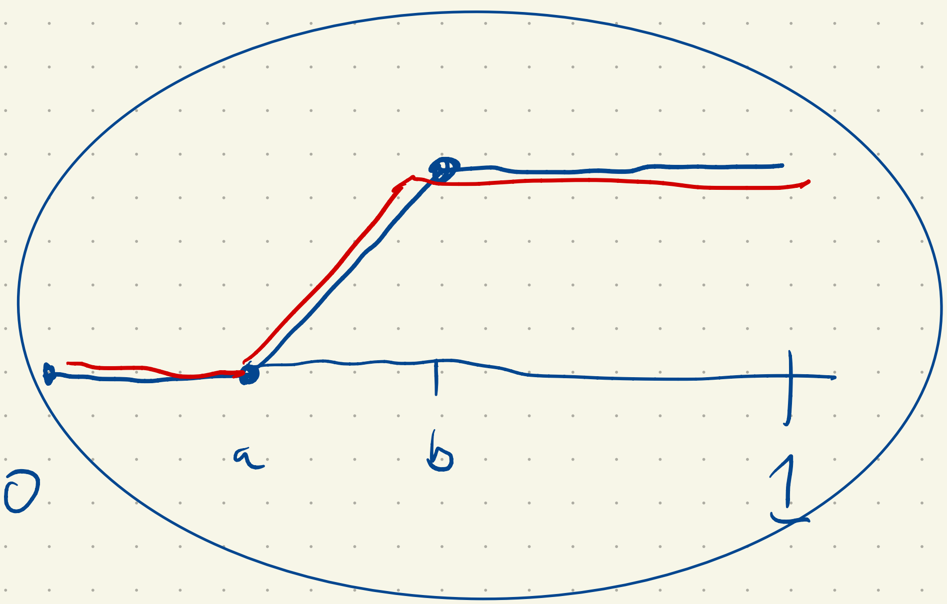
$\mathrm{abs}_a \in \overline{P[0,1]} \qquad \forall a \in [0,1]$

R_a



$R_a \in \overline{P[0,1]} \qquad R_a(x) = \frac{1}{2}\left(\underline{x - a} + \mathrm{abs}_a(x) \right)$

$V_{a,b} \qquad 0 \le a < b \le 1$



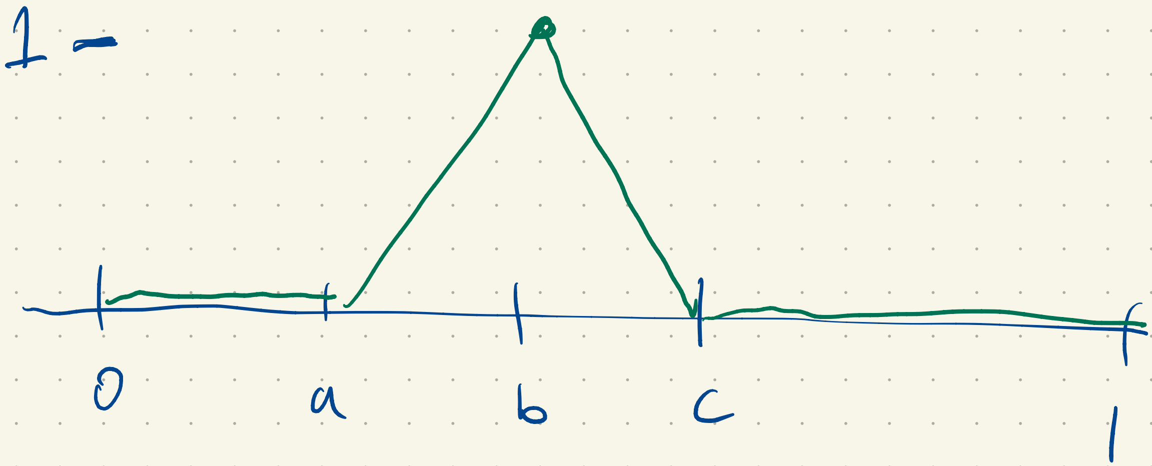
$$V_{a,b}(x) = \frac{R_a(x) - R_b(x)}{b-a}$$

$$V_{a,b} \in \overline{P[0,1]}$$

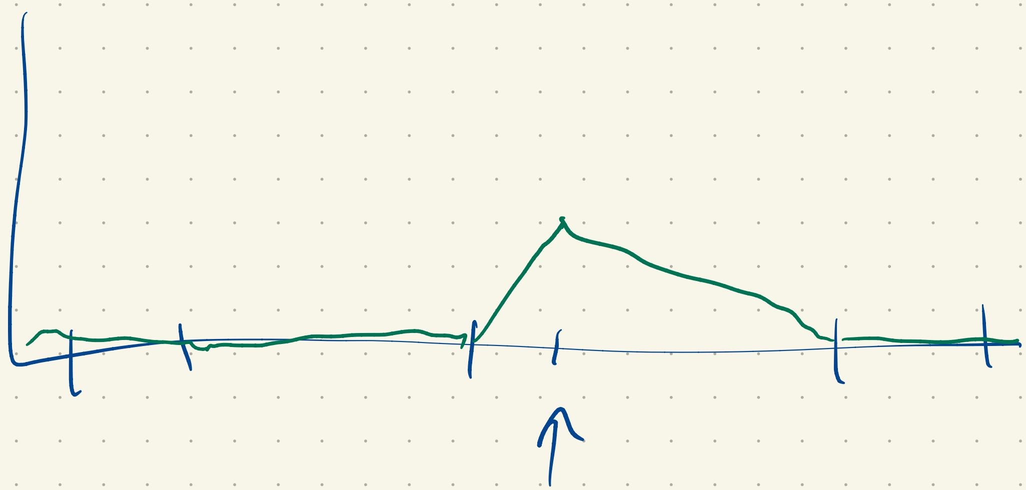
"hat"

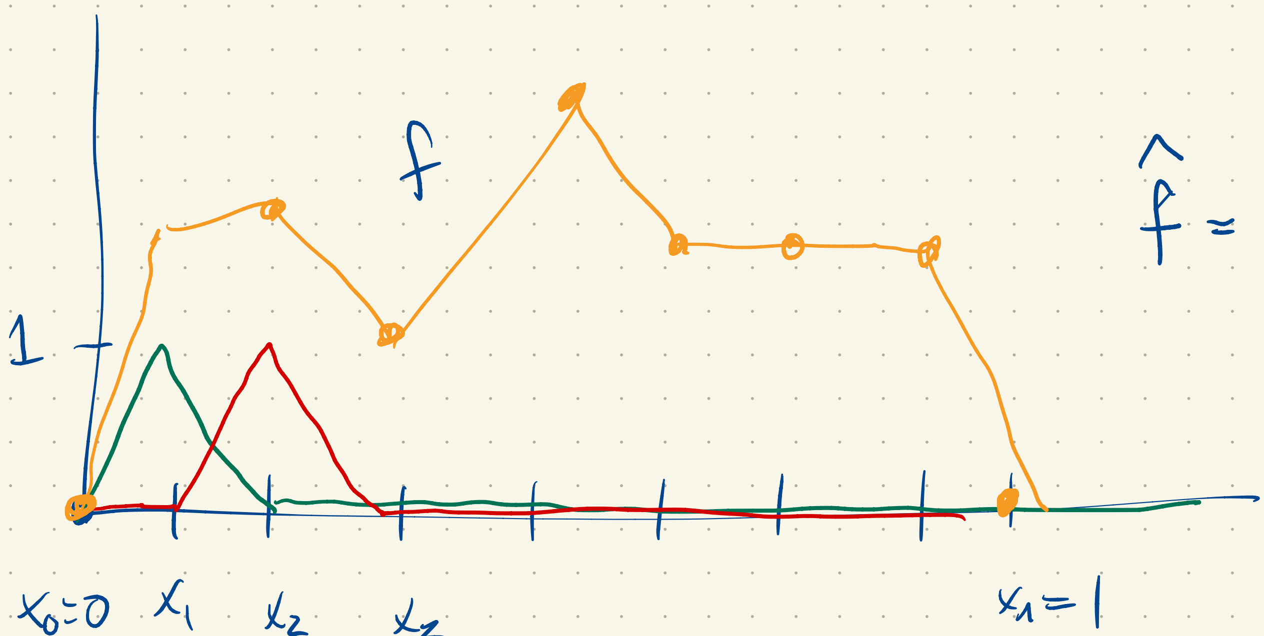
$H_{a,b,c}$ $\quad 0 \le a < b < c \le 1$

$$H_{a,b,c} = V_{a,b} - V_{b,c}$$



$$H_{a,b,c} \in \overline{P[0,1]}$$





$$\hat{f} = \sum_{k=1}^{n-1} a_k H_k$$

$$= \sum_{k=1}^{n-1} f(x_k) H_k$$

$$H_1, H_2, \ldots, H_{n-1} \in \overline{P[0,1]}$$

$$H_k(x_\ell) = \begin{cases} 1 & k=\ell \\ 0 & k\neq\ell \end{cases}$$