Weierstrass-M test

Used to detect uniform convergence of series of functions.

(i.e. that the partial sums converge uniformly)

Thm: Suppose (f_n) is a sequence of functions from a set X to $\mathbb{R}$ and for each n there exists $M_n \geq 0$ such that for all $x \in X$, $|f_n(x)| \leq M_n$.

If $\sum_{n=1}^{\infty} M_n$ converges then $\sum_{n=1}^{\infty} f_n$ converges uniformly to a limit f.

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \sin(nx) \quad \text{on } \mathbb{R}$$

$$\left| \frac{1}{2^n} \sin(nx) \right| \le \frac{1}{2^n}$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

$$\int_a^b \sum_{n=1}^{\infty} \frac{1}{2^n} \sin(x) \, dx = \sum_{n=1}^{\infty} \int_a^b \frac{1}{2^n} \sin(nx) \, dx$$

Pf: Let $s_n = \sum_{k=1}^{n} f_k$. Then if $n > m$ and $x \in X$

$$|s_n(x) - s_m(x)| = \left| \sum_{k=m+1}^{n} f_k(x) \right|$$

$$\leq \sum_{k=m+1}^{n} |f_k(x)|$$

$$\leq \sum_{k=m+1}^{n} M_k .$$

Given $\varepsilon > 0$ we can find N so if $n > m \geq N$

$\sum_{k=m+1}^{n} M_k < \varepsilon$ and consequently $\|s_n - s_m\|_\infty < \varepsilon$ also.

Hence (s_n) is Cauchy in $B(X)$ and hence

converges. $\square$

Shorter proof:

Since $B(X)$ is complete every absolutely convergent series in $B(X)$ converges.

Note $\|f_n\|_\infty \leq M_n$.

So $\sum_{k=1}^{\infty} \|f_k\|_\infty \leq \sum_{k=1}^{\infty} M_k$ and converges. Hence $\sum_{k=1}^{\infty} f_k$ converges.

Application: Power series.

$$\sum_{k=0}^{\infty} \underbrace{r^k}_{b_k}$$ converges so long as $|r|<1$.

$$\sum_{k=1}^{\infty} k\, r^{k-1} \qquad \sum_{k=0}^{\infty} \underbrace{(k+1)\, r^k}_{a_k} \qquad \sum_{k=0}^{\infty} a_k$$

$$\frac{b_k}{b_{k+1}} \qquad \frac{r^k}{r^{k+1}} \qquad \frac{1}{r}$$

$$\boxed{\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| < 1}$$

$$\frac{a_{k+1}}{a_k} = \frac{(k+2)\, r^{k+1}}{(k+1)\, r^k} = \boxed{\frac{(k+2)}{(k+1)}}\, r \longrightarrow r$$

Power Series:

Suppose $\sum_{k=0}^{\infty} a_k x^k$ converges for some $x_0 \in \mathbb{R}$, $x_0 \neq 0$.

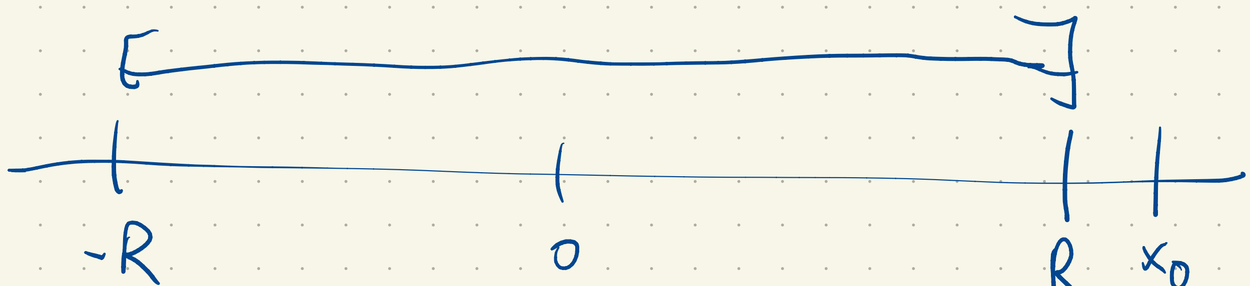
Let $R \in (0, |x_0|)$ $(0 < R < |x_0|)$.

Then the series converges uniformly on $[-R, R]$.

Moreover the limit f is differentiable on $[-R, R]$

and $f'(x) = \sum_{k=1}^{\infty} k\, a_k\, x^{k-1}$ for all $x \in [-R, R]$

and the convergence is uniform.



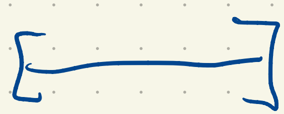
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\sin(x) = 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

$$\cos(x) = 1 + 0 - \frac{x^2}{2!} + 0 + \frac{x^4}{4!} - \cdots$$

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$



$-R$ $\quad$ R

$$f'(x) = \sum_{k=0}^{\infty} k a_k x^{k-1}$$

$[-R', R']$ $\quad$ $0 < R' < R$

$$f''(x) = \sum_{k=0}^{\infty} k(k-1) x^{k-2}$$

$$\sum_{k=1}^{\infty} \frac{x^k}{k}$$

$x = -1$

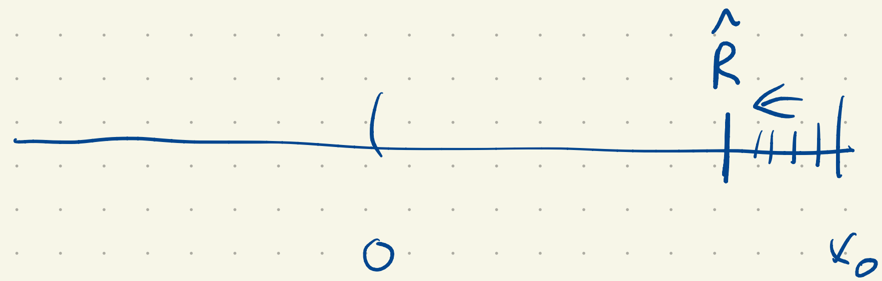
$x_0 = -1$

$[-1, 1]$

$[-1+\varepsilon, 1-\varepsilon] \leftarrow$

R_n

$f^{(n)}$



$[-\hat{R}, \hat{R}] \leftarrow$ all derivatives converge uniformly here.

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

$$0^0 = 1$$

$$f(0) = a_0$$

$$f'(x) = \sum_{k=0}^{\infty} k a_k x^{k-1}$$

$$a_0 + a_1 x + a_2 x^2 + \cdots$$

$$0 + a_1 + 2 a_2 x^1 + 3 a_3 x^2$$

$$f'(0) = a_1$$

$$f''(x) = 2 a_2 + 3 \cdot 2 a_3 x + \cdots$$

$$f'''(x) = 0 + 3 \cdot 2 a_3 + \cdots$$

$$f''(0) = 2 a_2$$

$$f'''(0) = 3! \, a_3$$

$$f^{(4)}(0) = 4! \, a_4$$

$$a_k = \frac{f^{(k)}(0)}{k!}$$

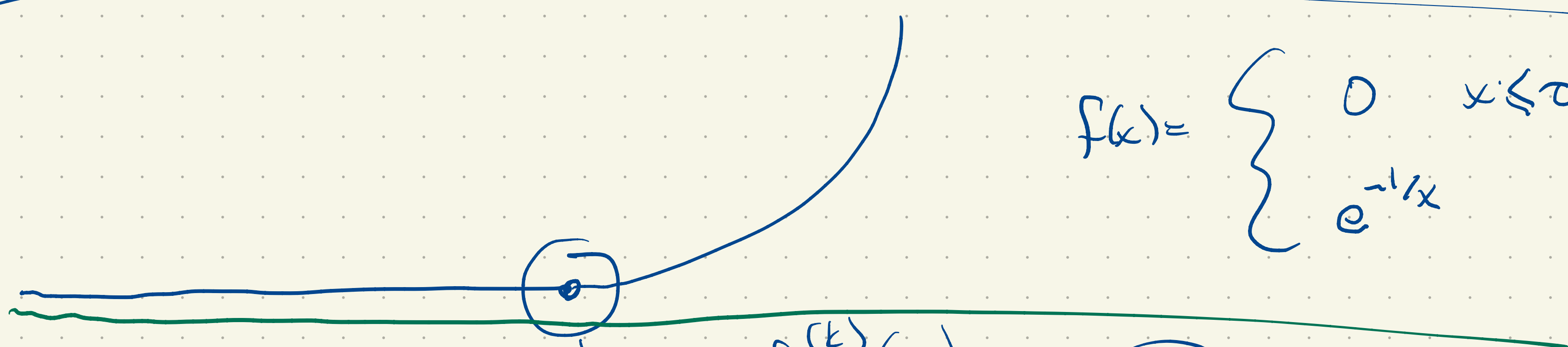
e^x $\qquad a_k = \dfrac{1}{k!}$

$f(x)$, smooth on all of $\mathbb{R}$

$f^{(k)}(0) = 0$

$\Rightarrow f(x) = 0 \ \forall x$?

$$a_k = \frac{f^{(k)}(0)}{k!}$$



$f^{(k)}(0) = 0$

$$f(x) = \begin{cases} 0 & x \leq 0 \\ e^{-1/x} & \end{cases}$$

Pf: Consider the numbers $a_k x_0^k$.

Since $\sum_{k=0}^{\infty} a_k x_0^k$ converges $a_k x_0^k \to 0$ and

we can find M so $|a_k x_0^k| \leq M$ for all k.

Let $f_k(x) = a_k x^k$ on $[-R, R]$.

Then $$|f_k(x)| = |a_k x^k| = |a_k x_0^k| \cdot \left|\frac{x}{x_0}\right|^k$$

$$\leq M \left|\frac{x}{x_0}\right|^k$$

$$\leq \underbrace{M \left(\frac{R}{|x_0|}\right)^k}_{} \longrightarrow M_k$$

Let $M_k = M\left(\frac{R}{|x_0|}\right)^k$. Since $|x_0| > R$,

$\sum_{k=0}^{\infty} M_k = \sum_{k=0}^{\infty} M\left(\frac{R}{|x_0|}\right)^k$ converges. By the Weierstrass M-test

$\sum_{k=0}^{\infty} a_k x^k$ converges uniformly on $[-R, R]$.

Moreover $f_k'(x) = k a_k x^{k-1}$ and

$$|f_k'(x)| = |k a_k x^{k-1}|$$

$$= k \left| a_k x_0^k \frac{x^{k-1}}{x_0^k} \right|$$

$$= \frac{k}{|x_0|} \left|a_k x_0^k\right| \left|\frac{x^{k-1}}{x_0^{k-1}}\right|$$

$$\leq \frac{M}{|x_0|} k \left|\frac{x}{x_0}\right|^{k-1}$$

$$\leq \frac{M}{|x_0|} k \left(\frac{R}{|x_0|}\right)^{k-1}.$$

$$\sum_{k=1}^{\infty} k r^{k-1} \qquad |r|<1$$

Since $\sum_{k=1}^{\infty} \frac{M}{|x_0|} k \left(\frac{R}{|x_0|}\right)^{k-1}$ converges and

$\frac{R}{|x_0|} < 1$

the Weierstrass M-test implies

$$\sum_{k=1}^{\infty} k\, a_k\, x^{k-1} \quad \text{converges uniformly on } [-R, R].$$

$$\left.\begin{array}{l} \displaystyle\sum_{k=0}^{\infty} f_k(x) \xrightarrow{\text{unif.}} f \\[2ex] \displaystyle\sum_{k=0}^{\infty} f_k'(x) \xrightarrow{\text{unif.}} g \end{array}\right] \text{ on } [-R,R] \Rightarrow f'(x) = g(x) \text{ on } [-R,R]$$

By our earlier result on derivatives and uniform convergence, $f(x) = \sum_{k=0}^{\infty} a_k x^k$ is diff on $[-R, R]$

and $f'(x) = \sum_{k=1}^{\infty} k\, a_k\, x^{k-1}$.

$\square$