Prop: Suppose (f_n) is a seq of functions on $[a,b]$ such that

1) Each f_n is continuous and differentiable on $[a,b]$
2) Each f_n' is continuous on $[a,b]$ $(*)$
3) $f_n' \to g$ uniformly for some g
4) $f_n(x_0) \to c$ for some $x_0 \in [a,b]$

Then there exists a differentiable function f on $[a,b]$ such that

1) $f_n \to f$ uniformly

2) $f' = g$

3) $f(x_0) = c$

$f_n \to f$ uniformly

$f_n' \to g$ unif.

$f' = g$

$$\left(\lim_n f_n\right)' = \left(\lim_n f_n'\right)$$

Pf: Observe for each n,

$$f_n(x) = f_n(x_0) + \int_{x_0}^{x} f_n'(s)\, ds$$

by the FTC (using continuity of f_n').

Hence for any $x \in [a,b]$

$$f_n(x) \to c + \int_{x_0}^{x} g(s)\, ds,$$

where we have used uniform convergence of f_n's to g.

Let $f(x) = c + \int_{x_0}^{x} g(s)\, ds$.

We have shown $f_n \to f$ pointwise.

Note that $f(x_0) = c$ and by the FTC and continuity of g, $f' = g$.

Moreover, for any $x \in [a,b]$

$$|f_n(x) - f(x)| \leq |f_n(a) - f(a)| + \int_a^x |f_n'(s) - g(s)|\, ds$$

$$\leq |f_n(a) - f(a)| + (b-a)\|f_n' - g\|_\infty,$$

Given $\varepsilon > 0$ we can find N so that the RHS of this inequality is less than ε if $n \geq N$. This holds for all $x \in [a,b]$

so $f_n \to f$ uniformly.

$\square$

X set

$B(X)$ $\{f : X \to \mathbb{R} : f$ is bounded$\}$

($\exists M,$ $|f(x)| \le M \ \forall x \in X$)

Exercise: $\|\cdot\|_\infty$ is a norm on $B(X)$ (which is a vector space).

Exercise $f_n \to f$ in $B(X) \iff f_n \to f$ uniformly

Prop: $B(X)$ is complete. l_∞ is complete.

Sketch: Let (f_n) be Cauchy in $B(X)$.

a) candidate

$x \in X$ $\quad |f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty$

$\forall x \in X \; \exists \; f(x), \quad f_n(x) \to f(x)$.

b) Show $f \in B(X)$

Use Cauchy sequences are bounded.

c) Show uniform convergence.

Let $\varepsilon > 0$. Pick N so $n, m \geq N \Rightarrow \|f_n - f_m\|_\infty \leq \varepsilon$.

Then, if $n \geq N$ and $x \in X$,

$$|f(x) - f_n(x)| = \lim_{m \to \infty} \underbrace{|f_m(x) - f_n(x)|}_{< \varepsilon}$$

$$\leq \varepsilon$$

$f_n \to f$ uniformly,

$\|f - f_n\|_\infty \leq \varepsilon$ if $n \geq N$. $f_n \xrightarrow{\|\cdot\|_\infty} f$.

Cor: $C[0,1]$ is complete.

Pf: Note $C[0,1] \subseteq B([0,1])$ with the same norm.

So it suffices to show $C[0,1]$ is closed.

Let (f_n) be a sequence in $C[0,1]$ converging to some $f \in B([0,1])$. [Job: $f \in C[0,1]$]

Since the f_n's converge uniformly, f is continuous.

More generally, if X is a metric space

$$C_b(X) = C(X) \cap B(X)$$

Exercise: Use the same argument to show $C_b(X)$ is complete.

Exercise: If X is compact then $C(X) \subseteq \mathcal{B}(X)$ and $C(X)$ is complete.