$\forall x \in X \quad f_n(x) \longrightarrow f(x)$

pointwise convergence

$$\lim_{x \to x_0} f(x) = f(x_0)$$

$$\lim_{x \to x_0} \lim_{n \to \infty} f_n(x)$$

does not not preserve continuity

$$= \lim_{n \to \infty} \underbrace{\lim_{x \to x_0} f_n(x)}$$

$$= \lim_{n \to \infty} f_n(x_0)$$

$$= f(x_0)$$

$$\lim \int f_n \not\longrightarrow \int \lim f_n = \int f$$

$$\lim f_n' \not\longrightarrow (\lim f_n)' = f'$$

Def: A sequence (f_n) of functions converges uniformly to a function f ($f_n \rightrightarrows f$, $f_n \to f$ uniformly)

if for all $\varepsilon > 0$ there exists N so if $n \geq N$

then for all $x \in X$ $\quad d_Y(f(x), f_n(x)) < \varepsilon$.

$$f_n : X \to Y$$

set $\qquad$ metric space

$$f_n(x) = x^n \quad \text{on } [0,1]$$

$f_n \to f$ uniformly?

No!

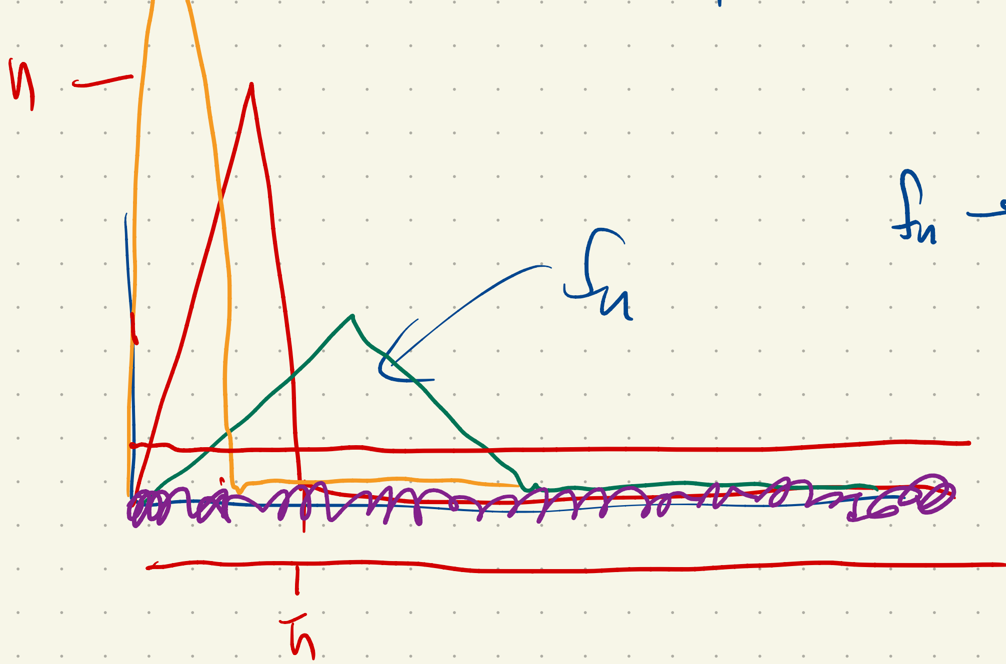
$\epsilon = \frac{1}{4}$

$N \quad n \geq N, \quad |f(x) - f_n(x)| < \frac{1}{4}$

$\forall x \in [0,1]$

$\forall n \quad \exists x \in [0,1]$

with $f_n(x) = \frac{1}{2}$ by the IVT.



$f_n \to 0$ pointwise.

$f_n \to 0$ uniformly?

No.

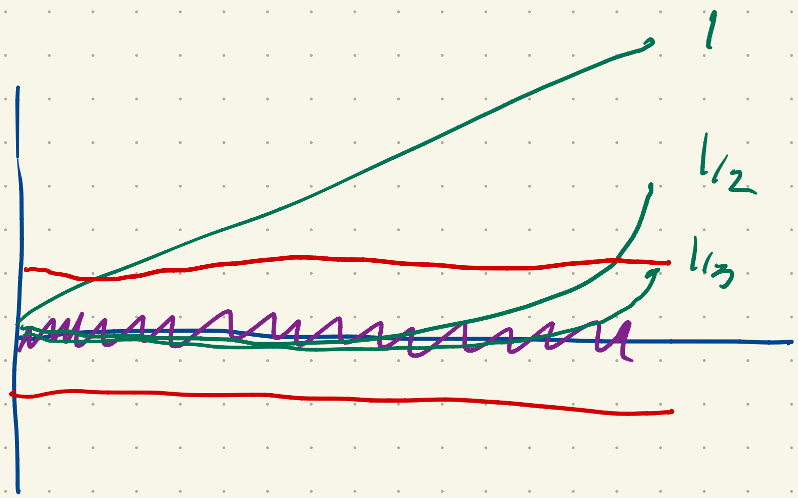
$f_n(x) = \frac{1}{n} x^n \longrightarrow 0$ pointwise

$f(x) = 0$

$f_n'(x) = x^{n-1}$ $\qquad f'(x) = 0$

$f_n'(1) = 1$ $\qquad f'(1) = 0$

is this uniform convergence? $\longrightarrow$ Yes!



$\epsilon > 0$

$N \qquad \frac{1}{N} < \epsilon$

$n \geq N \quad |f_n(x) - 0| \overset{?}{<} \epsilon$

$|f_n(x)| \leq \frac{1}{n}$

Uniform convergence plays well with continuity and integration.

Prop: Suppose $f_n : X \to Y$ are all continuous at $x_0 \in X$ and converge uniformly to a limit f. Then f is continuous at x_0.

Cor: If (f_n) is a sequence of continuous functions converging uniformly to a limit f, the limit is continuous.

Pf: Let $\varepsilon > 0$. There exists N so if $n \geq N$, $d_Y(f(x_0), f_n(x_0)) < \varepsilon$. Since f_N is continuous at x_0 there exists $\delta > 0$ so if $d_X(x, x_0) < \delta$ then $d(f_N(x_0), f_N(x)) < \varepsilon$.

But then if $d_X(x, x_0) < \delta$,

$$d_Y(f(x), f(x_0)) \le d_Y(f(x), f_N(x)) + d_Y(f_N(x), f_N(x_0)) + d_Y(f_N(x_0), f(x_0))$$

$$< \underbrace{\varepsilon}_{\text{u.c.}} + \underbrace{\varepsilon}_{\text{cont. of } f_N} + \underbrace{\varepsilon}_{\text{u.c.}}$$

$$= 3\varepsilon.$$

$\square$

"The uniform limit of continuous functions is continuous."

Integration:

Baby version,

Riemann int? Yes!

Prop: Suppose (f_n) is a sequence of continuous functions on $[a,b]$ and $f_n \to f$ uniformly. Then f is Riemann integrable and

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f .$$

Pf: Since $f_n \to f$ uniformly, f is continuous and is hence Riemann integrable. Let $\varepsilon > 0$ and pick N so if $n \geq N$, $|f(x) - f_n(x)| < \varepsilon$ for all $x \in [a,b]$.

Then if $n \geq N$,

$$\left| \int_a^b f_n - \int_a^b f \right| = \left| \int_a^b (f_n - f) \right|$$

$$\leq \int_a^b \underbrace{|f_n - f|}$$

$$\leq \int_a^b \varepsilon$$

$$= (b-a)\varepsilon .$$

So $\lim_{n \to \infty} \int_a^b f_n = \int_a^b f$. $\square$

$\lim_{n \to \infty} z_n = z$ $\quad z_n \in \mathbb{R}$ $\quad n \geq N$ $\quad \underline{|z - z_n| < \varepsilon}$

$n \geq N$

Regarding differentiation uniform convergence is not enough.

$$f_n(x) = \frac{1}{n} x^n$$

$$f_n(x) = \frac{1}{n}\sin(nx) \quad \text{on} \quad [0, \pi]$$

$$f_n \to 0 \quad \text{uniformly}$$

$$f_n'(x) = \cos(nx)$$