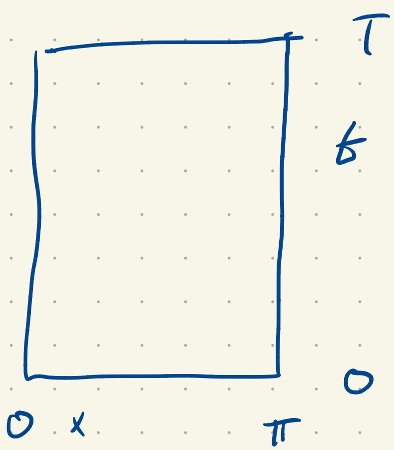
The same inequality is obvious if $x = 0$.

Exercise: Any finite dim vector space has the property that all norms are equivalent.

Show that if $T: \mathbb{R}^n \to V$ is a linear iso then $x \longmapsto \|Tx\|_V$ is a norm on $\mathbb{R}^n$!

Sequences and series of functions



Want to solve

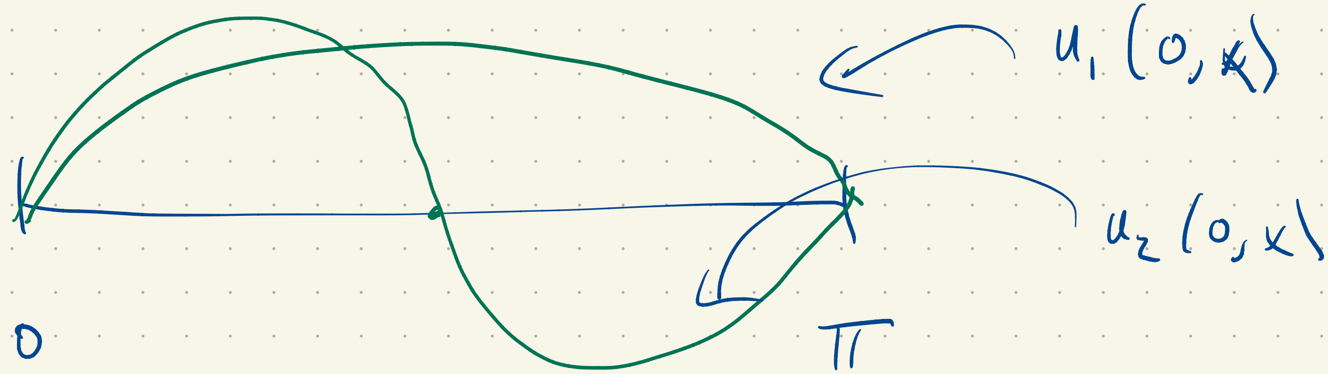
$$u_t = u_{xx}$$

$$u\big|_{x=0} = u\big|_{x=\pi} = 0$$

$$u(x, 0) = u_0(x)$$

Some solutions $\quad u_k(t, x) = e^{-k^2 t} \sin(kx) \quad k \in \mathbb{Z}$

$$u_k(0, x) = \sin(kx)$$



And: $\quad u = \sum_{k=1}^{N} c_k u_k \qquad$ (c_k's constants)

These also solve.

Y'all are bold.

$$u = \sum_{k=1}^{\infty} c_k u_k$$

What does this even mean?

Maybe for all t, x

$$u(t,x) = \sum_{k=1}^{\infty} c_k u_k(t,x)$$

Is it true: $\partial_t u = \partial_x^2 u$

$$\partial_t u(t,x) = \partial_t \sum_{k=1}^{\infty} a_k u_k(t,x)$$

$$\stackrel{?}{=} \sum_{k=1}^{\infty} a_k \partial_t u_k(t,x)$$

$$u(0,x) = \sum_{k=1}^{\infty} c_k \sin(kx)$$

How to determine c_k's.

Given $u_0(x)$, how to determine c_k's?

$$u_0(x) = \sum_{k=1}^{N} c_k \sin(kx)$$

$$\int_0^{\pi} \sin(kx)\sin(\ell x)\,dx = \begin{cases} 0 & k \neq \ell \\ \frac{\pi}{2} & k = \ell \end{cases}$$

$$\int_0^{\pi} \sin^2(kx)\,dx = \int_0^{\pi} \cos^2(kx)\,dx$$

$$\int_0^{\pi} u_0(x)\sin(\ell x)\,dx = \int_0^{\pi} \left[\sum_{k=1}^{N} c_k \sin(kx)\right] \sin(\ell x)\,dx$$

$$= \sum_{k=1}^{N} c_k \int_0^{\pi} \sin(kx)\sin(\ell x)\,dx$$

$$= c_\ell \frac{\pi}{2}$$

$$c_\ell = \frac{2}{\pi}\int_0^\pi u_0(x)\sin(\ell x)\,dx$$

Does this work more generally?

$u_0(x)$, arbitrary, cts

Define c_ℓ via

Is it true that $u_0(x) = \sum_{k=1}^{\infty} c_k \sin(kx)$??

$$\int_0^\pi u_0(x)\sin(\ell x)\,dx = \int_0^\pi \left[\sum_{k=1}^{\infty} c_k \sin(kx)\right]\sin(\ell x)\,dx$$

$$\overset{?}{=} \sum_{k=1}^{\infty} c_k \int_0^\pi \sin(kx)\sin(\ell x)\,dx$$

$$= c_0 \frac{\pi}{2}$$

One is tempted from experience with power series that everything works out for the best.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\frac{d}{dx} e^x = \frac{d}{dx} \sum_{n=0}^{\infty} \frac{x^n}{n!} \overset{?}{=} \sum_{n=0}^{\infty} \frac{d}{dx} \frac{x^n}{n!}$$

$$= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

$$\int_0^1 e^x\,dx = \int_0^1 \sum_{n=0}^{\infty} \frac{1}{n!} x^n\,dx$$

$$\stackrel{?}{=} \sum_{n=0}^{\infty} \int_0^1 \frac{1}{n!} x^n\,dx$$

$$= \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!}\Big|_0^1$$

$$= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} = \sum_{n=1}^{\infty} \frac{1}{n!} = e^1 - 1 \quad \checkmark$$

We'll translate our concerns into questions about sequences & functions.

$$f_N(x) = \sum_{k=0}^{N} \frac{1}{k!} x^k \qquad f(x) = e^x$$

$$f(x) = \lim_{N\to\infty} f_N(x) \quad \forall\, x \in \mathbb{R}$$

Is it true that $\partial_x f(x) = \lim_{N\to\infty} \partial_x f_N(x)$?

Is it true that $\int_0^1 f(x)\,dx = \lim_{N\to\infty} \int_0^1 f_N(x)\,dx$

Sadly, things don't always work out.

Def: Let (f_n) be a sequence of functions from a set A to a metric space Y. We say the sequence *converges pointwise* to $f: A \to Y$ if for all $a \in A$, $f_n(a) \xrightarrow{Y} f(a)$.

That is, $\forall a \in A$ $\lim_{n\to\infty} f_n(a) = f(a)$.

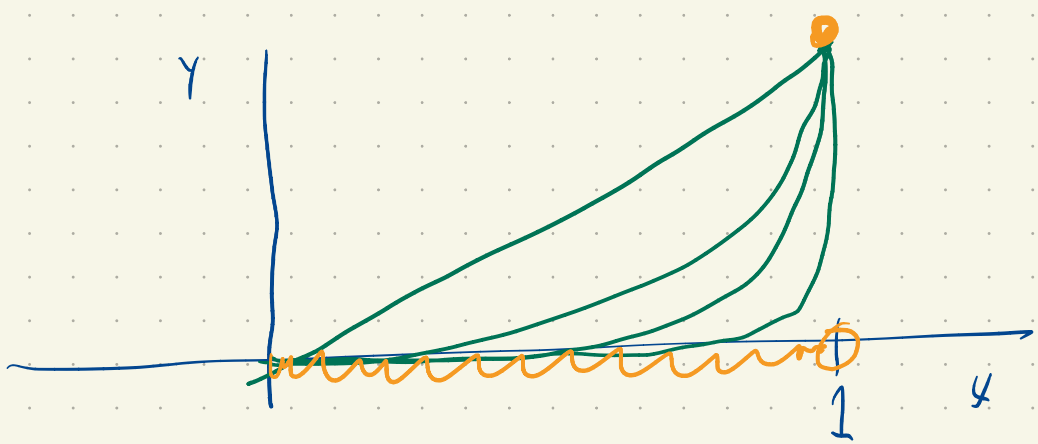
Grim news.

1) The pointwise limit of continuous functions need not be continuous.

$A = [0,1] \quad Y = \mathbb{R}$

$f_n(x) = x^n$

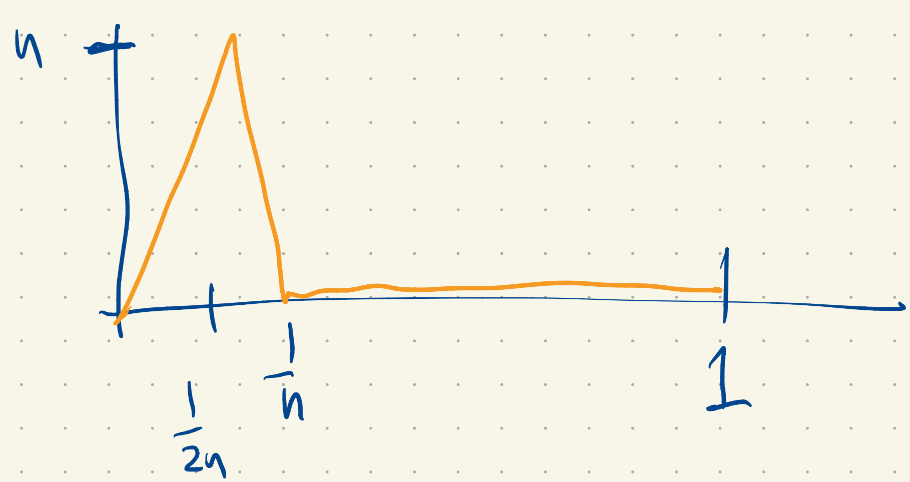
$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & x \neq 1 \\ 1 & x = 1 \end{cases}$$



2) The pointwise limit of Riemann integrable functions need not be Riemann integrable. In the event that it is, it need not be the case that

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx \qquad \left(f_n \to f \text{ pointwise}\right)$$

f_n



$f_n : [0,1] \to \mathbb{R}$

$f_n(0) = 0$

$f_n \to 0$ pointwise

$\uparrow$

f

$$\int_0^1 f_n(x)\,dx = \frac{1}{2} \text{ for all } n.$$

$$\int_0^1 f(x)\,dx = 0$$

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 f(x)\,dx$$ ☹

3) If $f_n \to f$ pointwise and the f_n's are differentiable, it need not be the case that f is differentiable. In the event that it is, it need not be the case that

$$\lim_{n\to\infty} f_n'(x) = f'(x)$$

$$f_n(x) = \frac{1}{n} x^n \quad \text{on} \quad [0,1]$$

$f_n \to 0$ f pointwise

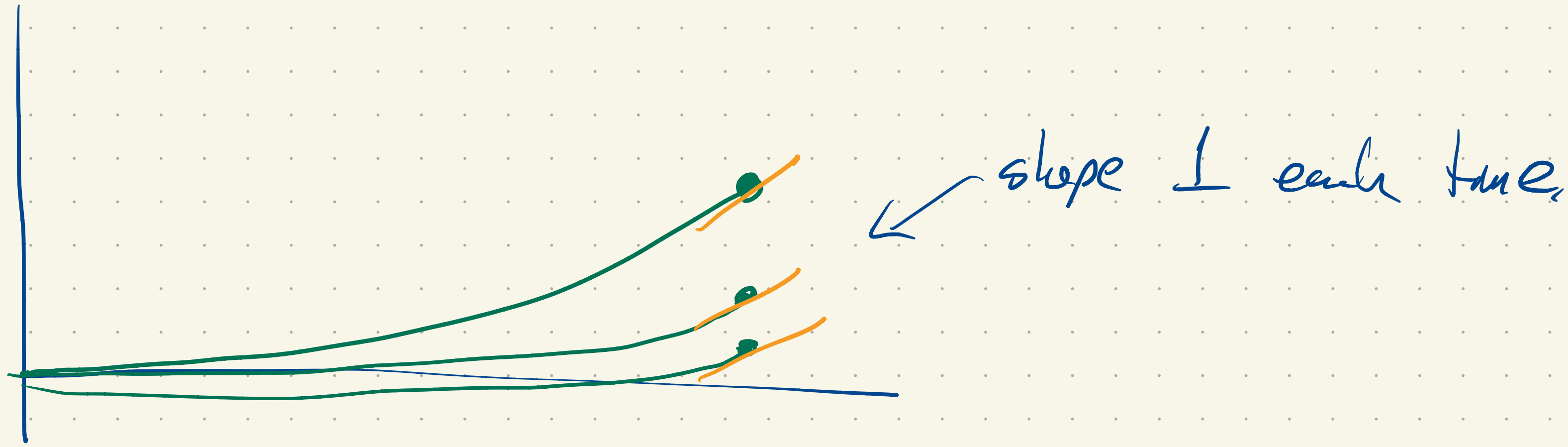
$$\lim_{n\to\infty} f_n'(1) \neq f'(1)$$

$$f_n'(x) = x^{n-1} \qquad f_n'(1) = 1$$

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases} \quad \text{on } [0,1]$$

not Riemann Integrable

(r_n) $\mathbb{Q} \cap [0,1]$



There are many useful notions of convergence of functions.

Pointwise convergence is just one.

Def: Suppose (f_n) is a sequence of functions from a set A to a metric space Y. We say the sequence converges uniformly to a limit $f: A \to Y$ if for all $\varepsilon > 0$ there exists N so if $n \geq N$, $d_Y(f_n(x), f(x)) < \varepsilon$ for all $x \in X$.

Pointwise: N depends on x

Uniform: One N to rule them all (works for all x).