Last class:

$$T : X \to Y \text{ , linear}$$

T is cts $\iff \exists\, C$ s.t.

$$(*) \quad \|Tx\|_Y \leq C \|x\|_X \quad \forall x \in X$$

If $\hat{C} \geq C$ then it also works in

So one is interested in the least C that "works"

If $x \neq 0$ $(*)$ is rewritten

$$\frac{\|Tx\|_Y}{\|x\|_X} \leq C$$

$$\boxed{\sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Tx\|_Y}{\|x\|_X}} \leftarrow \text{if finite, is the smallest } C \text{ that works}$$

$\rightarrow \|T\|$, operator norm of T

$B(X,Y)$ is the set of all continuous (bounded) linear maps from X to Y.

Exercise: $B(X,Y)$ is a vector space and the operator norm is a norm on it.

Back to our story: when are two norms equivalent?

$$\underbrace{(X, \|\cdot\|_1)}_{X_1}, \quad \underbrace{(X, \|\cdot\|_2)}_{X_2}$$

Norms are equivalent iff

$$\left.\begin{array}{l} i: X_1 \to X_2 \\ i^{-1}: X_2 \to X_1 \end{array}\right. \quad \text{are cts.}$$

i: X_1 to X_2 is continuous iff there exists C such that $\|ix\|_{X_2} \leq C \|x\|_{X_1}$

$$\|x\|_{X_2} \le \overbrace{C_1}^{C} \|x\|_{X_1}$$

$$\|x\|_{X_1} \le C_2 \|x\|_{X_2} \longrightarrow \overbrace{\frac{1}{C_2}}^{c} \|x\|_{X_1} \le \|x\|_{X_2}$$

$$\boxed{c\|x\|_{X_1} \le \|x\|_{X_2} \le C\|x\|_{X_1}} \quad \forall x \in X$$

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on $\mathbb{R}^n$

$\|x\|_\infty \le \|x\|_1$ $\qquad$ $\|x\|_1 \le n \|x\|_\infty$

$\|x\|_\infty \le \|x\|_2$ $\qquad$ $\|x\|_2 \le \sqrt{n} \|x\|_\infty$

$\|x\|_2 \le \|x\|_1$ $\qquad$ $\|x\|_1 \le \sqrt{n} \|x\|_2$

CS - ineq.

On $\mathbb{R}^n$, the ℓ_1, ℓ_∞, ℓ_2 norms are all equivalent.

Claim: on $\mathbb{R}^n$ <u>all</u> norms are equivalent.

z

$\leftarrow \; l_1 \,,\; l_\infty$

$(\underbrace{1, \ldots, 1}_{n}, 0 \ldots 0)$

$\|z\|_1 \leq C \|z\|_\infty$

Lemma: A subset of $\mathbb{R}^n$ is compact w.r.t. l_1 norm iff it is closed and bounded.

$[-M, M] \times \cdots \times [-M, M] \leftarrow$ f.b.

level sets of continuous functions are closed.

Cor: The set $\{x \in \mathbb{R}^n : \|x\|_1 = 1\}$ is compact wrt l_1.

$x_n \to x \overset{?}{\Rightarrow} \|x\|_1 = 1$

$\|x_n\|_1 = 1$

$x_n \to x \overset{?}{\Rightarrow} \|x_n\| \to \|x\|$

Yeah, by the Δ ineq.

$$\|x\| = \|x - y + y\| \leq \|x-y\| + \|y\|$$

$$\|x\| - \|y\| \leq \|x-y\|$$

$$\|y\| - \|x\| \leq \|y-x\| = \|x-y\|$$

$$\|x\| - \|y\| \geq -\|x-y\|$$

$$-\|x-y\| \leq \|x\| - \|y\| \leq \|x-y\|$$

$$\boxed{\big|\, \|x\| - \|y\| \,\big| \leq \|x-y\|}$$

norm is Lip continuous with Lip constant 1.

$$d_{\mathbb{R}}(\|x\|, \|y\|) \leq 1 \cdot d_X(x,y)$$

Prop: Let $|||\cdot|||$ be a norm on $\mathbb{R}^n$. Then $|||\cdot|||$ is equivalent to the $\|\cdot\|_1$.

Cor: [Exercise] all norms on $\mathbb{R}^n$ are equivalent;

show equivalence of norms is an equivalence relation.

Pf: Let $e^{(k)} = (0, .., 1, 0 .. 0)$

k'th slot.

Let $C = \max_k (|||e^{(k)}|||)$.

$x = (c_1, c_2, ..., c_n)$

Then if $x \in \mathbb{R}^n$, $x = \sum_{k=1}^{n} c_k e^{(k)}$ and

$$|||x||| = \left|\left|\left| \sum_{k=1}^{n} c_k e^{(k)} \right|\right|\right|$$

$$\leq \sum_{k=1}^{n} |||c_k e^{(k)}|||$$

$$= \sum_{k=1}^{n} |c_k| \, |||e^{(k)}|||$$

$$\leq \sum_{k=1}^{n} C |c_k|$$

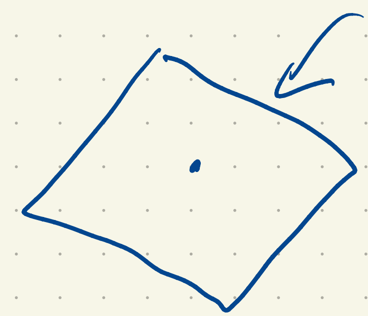
$$= C \, \|x\|_1 .$$

Conversely, to show the reverse inequality, consider $x, y \in \mathbb{R}^n$.

Then

$$\left| |||x||| - |||y||| \right| \leq |||x-y||| \leq C \, \|x-y\|_1 .$$

Hence the map $x \longmapsto x$ is Lip continuous from $(\mathbb{R}^n, \ell_1)$ to $(\mathbb{R}^n, |||\cdot|||)$. The set $A = \{x \in \mathbb{R}^n : \|x\|_1 = 1\}$ is compact with respect to ℓ_1 and hence $|||\cdot|||$ achieves a minimum c on A. Moreover $c > 0$.



Now consider $x \in \mathbb{R}^n$, $x \neq 0$. Then

$x/\|x\|_1 \in A$ and hence

$$|||\, x/\|x\|_1 \,||| \geq c \quad \text{and equivalently}$$

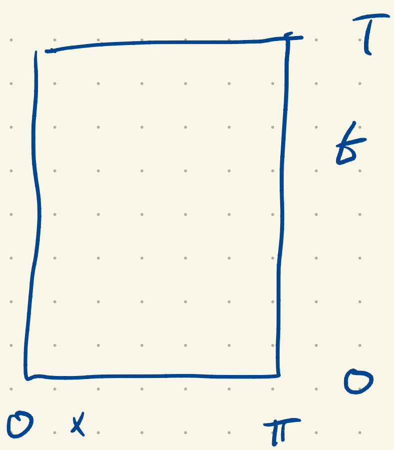
$$|||x||| \geq c \, \|x\|_1 .$$

The same inequality is obvious if $x = 0$.

Exercise: Any finite dim vector space has the property that all norms are equivalent.

Show that if $T: \mathbb{R}^n \to V$ is a linear iso then $x \longmapsto \|Tx\|_V$ is a norm on $\mathbb{R}^n$!

Sequences and series of functions



Want to solve

$$u_t = u_{xx}$$

$$u\big|_{x=0} = u\big|_{x=\pi} = 0$$