Recall: $\underbrace{(X, d_1)}_{X_1}$ $\underbrace{(X, d_2)}_{X_2}$

$$x_n \xrightarrow{d_1} x \iff x_n \xrightarrow{d_2} x$$

$$\implies$$

Metrics are equivalent if they determine the same convergent sequences.

$$X_1 \xrightarrow{i} X_2$$

$$i(x) \longrightarrow x$$

$i(x_n) \xrightarrow{d_2} i(x)$

$\forall$ sequences with $x_n \xrightarrow{d_1} x \Rightarrow \underbrace{x_n \xrightarrow{d_2} x}$ is same as

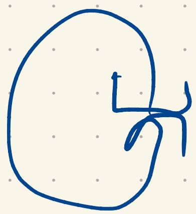
$i : X_1 \to X_2$ is continuous.

Metrics are equivalent iff

$$i : X_1 \to X_2$$

$$i^{-1} : X_2 \to X_1 \quad \text{are continuous.}$$

[A continuous map with a continuous inverse is known as a homeomorphism]



In the context of normed vector spaces:

$$X_1 = (X, \|\cdot\|_1) \qquad X_2 = (X, \|\cdot\|_2)$$

$$i : X_1 \to X_2$$

↪ is a linear map.

$$\begin{cases} i(x+y) = i(x) + i(y) \\ i(cx) = c\, i(x) \end{cases}$$

$\cancel{\delta_{x+y} = \delta_x + \delta_y}$

$$x+y = x+y$$

The equivalence of the metrics associated with the two norms is determined by the continuity of the <u>linear</u> map i.

Let's talk about continuity of linear maps.

Not all linear maps are continuous. (!)

$$P[0,1], \quad L_\infty$$

$$\partial : P[0,1] \to P[0,1]$$

$$\frac{d}{dx}\left(f(x)+g(x)\right) = f'(x)+g'(x)$$

$$f_n(x) = \frac{1}{n}x^n \qquad f_n \xrightarrow{L_\infty} 0 \quad (\text{in}$$

$$(\partial f_n)(x) = x^{n-1} \qquad \left\|\frac{1}{n}x^n\right\|_\infty = \frac{1}{n} \to 0$$

If ∂ were continuous, $\partial f_n \to 0$ (in L_∞)

$$(\partial f_n)(1) = 1 \quad \forall n,$$

$$\|\partial f_n - 0\|_\infty \geq 1$$

$\partial f_n \to 0$? No way!

$Z = \{$ sequences that end in a trail of 0's $\}$, ℓ_∞

$Z \overset{f}{\hookrightarrow} \ell_1$ ← not continuous.

$z \longmapsto z$

$$z_n = (\underbrace{\tfrac{1}{n}, \tfrac{1}{n}, \dots, \tfrac{1}{n}}_{n \text{ times}}, 0, 0, \dots)$$

$z_n \to z$
$\quad\quad\; \uparrow$
$\quad\quad\; 0$

$\|x\|_\infty = \sup_n |x_n|$

$\|z_n - 0\|_\infty = \frac{1}{n} \longrightarrow 0$

If f were continuous, $f(z_n) \to f(0)$ in ℓ_1

$z_n \to 0$ in ℓ_1

$$\| z_n - 0 \|_1 = \| z_n \|_1 = 1$$

Continuity of Linear Maps.

Lemma: Suppose $T: X \to Y$ is linear. Then T is continuous if and only if it is continuous at 0.

Pf: Evidently if T is continuous, it is continuous at 0.

Suppose T is continuous at 0. Suppose (x_n) is a sequence in X converging to some x. [Job: $T(x_n) \to T(x)$]

Since translation is continuous, $x_n - x \to 0$.

Since T is continuous at 0, $T(x_n - x) \to T(0) = 0$.

But by linearity, $T(x_n - x) = T(x_n) - T(x)$.

Again, by continuity of translation, $T(x_n) \to T(x)$. $\square$

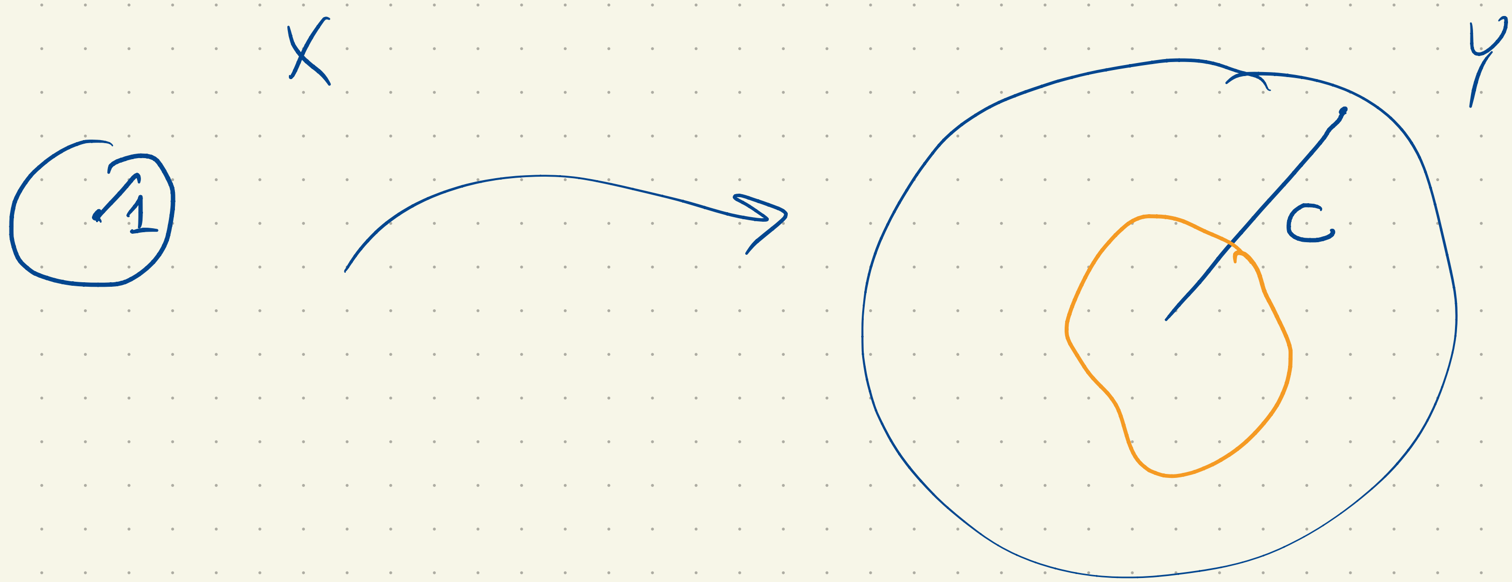
Continuity at 0 for linear maps:

Def: $T: X \to Y$ (linear) is **bounded** if there exists $C > 0$ such that

$$\|T(x)\|_Y \le C \|x\|_X \quad \text{for all } x \in X.$$

$C = 45$ $\quad x \in B_1^X(0)$ $\quad \|T(x)\|_Y \le 45$

$x \in B_2^X(0) \quad \|T(x)\|_Y \leq 90$



Prop: Suppose $T: X \to Y$ is linear. Then TFAE

1) T is bounded

2) $T(B_1^X(0))$ is a bounded subset of Y

3) T is continuous at 0.

Pf: 1) $\Rightarrow$ 2)

Suppose T is bounded with associated constant C.

Let $x \in B_1^X(0)$. Then $\|T(x)\|_Y \leq C\|x\|_X < C$.

Hence $T(B_1^X(0)) \subseteq B_C^Y(0)$ and is therefore bounded.

2) $\Rightarrow$ 1) Suppose $T(B_1^X(0)) \subseteq B_C^Y(0)$. Observe that for any $r > 0$ $T(B_r^X(0)) = rT(B_1^X(0))$

Consider some $x \neq 0$. Then $\frac{x}{2\|x\|_1} \in B_1^X(0)$ and

$$\left\| T\left(\frac{x}{2\|x\|_X}\right) \right\|_Y \leq C \quad \text{and} \quad \|T(x)\|_Y \leq 2C\|x\|_X.$$

This also holds for $x = 0$, trivially. So T is bounded.

2) $\Rightarrow$ 3) Suppose $T(B_1^X(0))$ is bounded and hence

contained in some $B_C^Y(0)$.

Let $\varepsilon > 0$. Pick $\delta = \varepsilon/C$.

If $\|x - 0\|_X < \delta$ then $x \in B_\delta^X(0)$ and

$$T(x) \in B_{\delta C}^Y(0) = B_\varepsilon^Y(0),$$

So $\|T(x) - T(0)\|_Y < \varepsilon$ and T is continuous at 0.

3) $\Rightarrow$ 2. Suppose T is continuous at 0.

Then there exists $\delta > 0$ so if $\|x - 0\|_X < \delta$,

$\|T(x) - T(0)\|_Y < 1$.

So $T(B_\delta^X(0)) \subseteq B_1^Y(0)$ and

$$T(B_1^X(0)) \subseteq B_{1/8}^Y(0).$$

Thus $T(B_1^X(0))$ is bounded in Y.

$$z_n = \left(\tfrac{1}{n}, \tfrac{1}{n}, \tfrac{1}{n}, \tfrac{1}{n}, \dots, \tfrac{1}{n}, 0, \text{---} \; 0\right) \in Z, \; \ell_\infty$$

$$\hat{z}_n = (\underbrace{1, 1, 1, \dots, 1}_{n}, 0, \dots \; 0)$$

$$\hat{z}_n \in B_2^Z(0).$$

$$f(\hat{z}_n) = (\underbrace{1, 1, 1, \dots, 1}_{n}, 0 \text{---} \text{---})$$

$$\|f(\hat{z}_n)\|_1 = n$$

Cor: Normed spaces X_1 and X_2 have equivalent metrics if and only if there exist constants c_1, c_2 with

$$\underbrace{c_1 \|x\|_2 \le \|x\|_1}_{} \;\; \underbrace{\le c_2 \|x\|_2}_{} \qquad \forall x \in X_1 = X_2.$$

$i^{-1} : X_2 \to X_1$ is cts

$$\|x\|_2 \le \frac{1}{c_1} \|x\|_1$$

$i : X_1 \to X_2$ is cts

$x \in \mathbb{R}^n$

$$\|x\|_\infty \leq \|x\|_1 \leq n\,\|x\|_\infty$$