Uniform Continuity

Def: A function $f: X \to Y$ is uniformly continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ so that if $x_1, x_2 \in X$ and $d(x_1, x_2) < \delta$ then $d(f(x_1), f(x_2)) < \varepsilon$.

[One δ works in all places all at once].

E.g. $\sin(x)$

Lipshitz functions

$< \varepsilon$

$K|x-y| < \varepsilon$

$|x-y| < \underbrace{\varepsilon/K}_{\delta}$

e.g. $f(x) = x^2$

$\hookrightarrow$ not U.C.

$\exists\ \varepsilon$ such that $\forall\ \delta > 0$ $\exists\ x_1, x_2,$ $d(x_1, x_2) < \delta$

$d(f(x_1), f(x_2)) \geq \varepsilon$.

$x_1 = x > 0$

$x_2 = x + h \qquad h > 0$

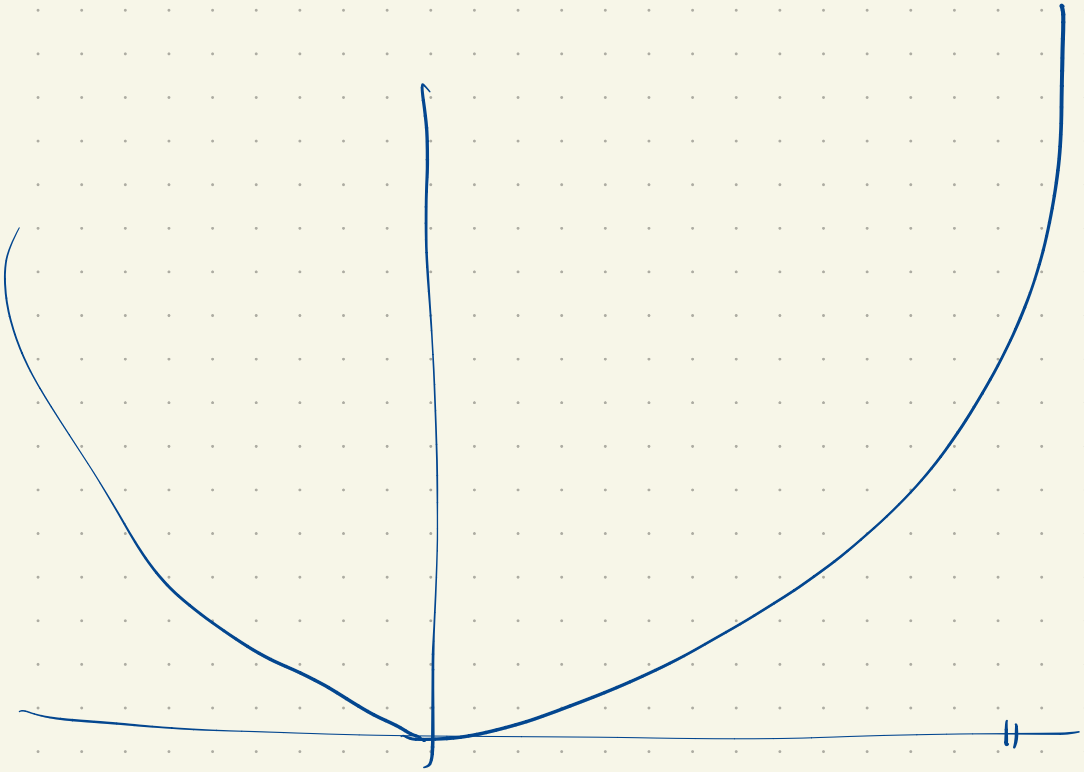
$f(x_2) - f(x_1) = 2xh + h^2$

$|f(x_2) - f(x_1)| = 2xh + h^2 \qquad \varepsilon = 1$

$\geq 2xh \qquad h < \delta$

$x > \frac{1}{2h}$

> 1

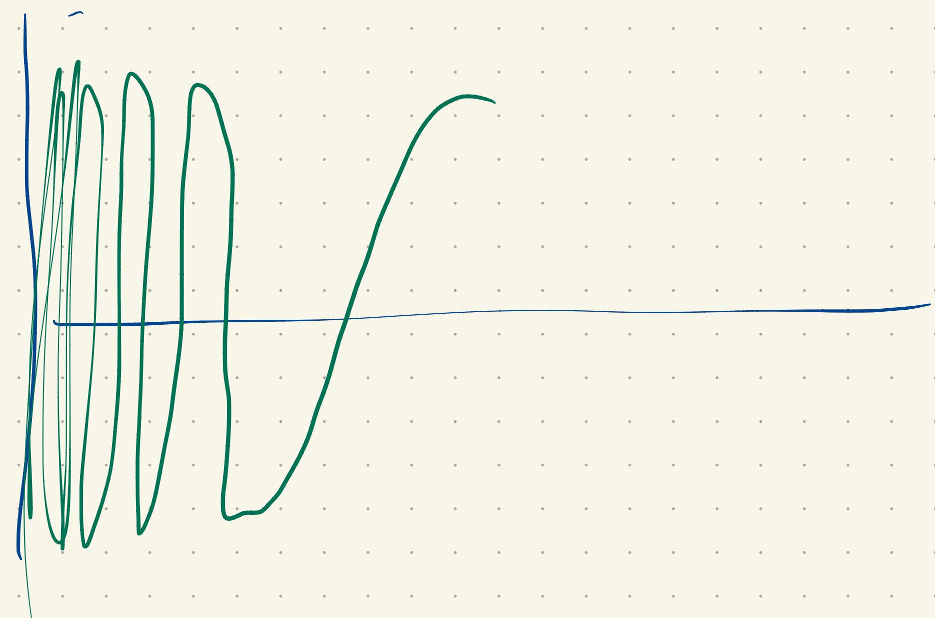


$f(x) = x^2$

$\hookrightarrow \quad \mathbb{R} \to \mathbb{R}$

$[0, \infty)$

$\sin(1/x)$ on $(0, 1]$



$\varepsilon = 1$

δ

Equivalent formulation of u.c.

$\forall \varepsilon > 0$, $\exists \delta > 0$ so $\forall x \in X$

$$f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$$

Exercise.

Prop: Suppose $f: X \to Y$ is uniformly continuous.

If $A \subseteq X$ is totally bounded then so is $f(A)$.

Pf: Let $A \subseteq X$ be totally bounded. Let $\varepsilon > 0$ and find $\delta > 0$ so that for all $x \in X$, $f(B_\delta^X(x)) \subseteq B_\varepsilon^Y(f(x))$.

Let $x_1, x_2, \ldots, x_n$ be a δ-net for A.

So $A \subseteq \bigcup_{k=1}^{n} B_\delta^X(x_k)$. But then

$$f(A) \subseteq f\Big(\bigcup_{k=1}^{n} B_\delta^X(x_k)\Big) \qquad \{f(x) : x \in A\}$$

$$= \bigcup_{k=1}^{n} f\big(B_\delta^X(x_k)\big)$$

$$\subseteq \bigcup_{k=1}^{n} B_\varepsilon^Y(f(x_k)).$$

So $f(x_1), \dots, f(x_n)$ is an ε-net for $f(A)$.

$\square$

Prop: Suppose X is compact and $f: X \to Y$ is continuous.

Then f is uniformly continuous.

$f(x) = x^2$ on $[0,1]$ is u.c.

Exercise: verify directly

Pf: Suppose to produce a contradiction that f is not uniformly continuous.

Then there exists an $\varepsilon > 0$ such that for all $n \in \mathbb{N}$ there exist $a_n, b_n \in X$ such that $d^X(a_n, b_n) < \frac{1}{n}$ but $d^Y(f(a_n), f(b_n)) \geq \varepsilon$.

Since X is compact we can extract a sequence (a_{n_k}) converging to some a.

Observe for each k $d(a, b_{n_k}) \leq d(a, a_{n_k}) + d(a_{n_k}, b_{n_k})$

$$\leq d(a, a_{n_k}) + \frac{1}{n_k}.$$

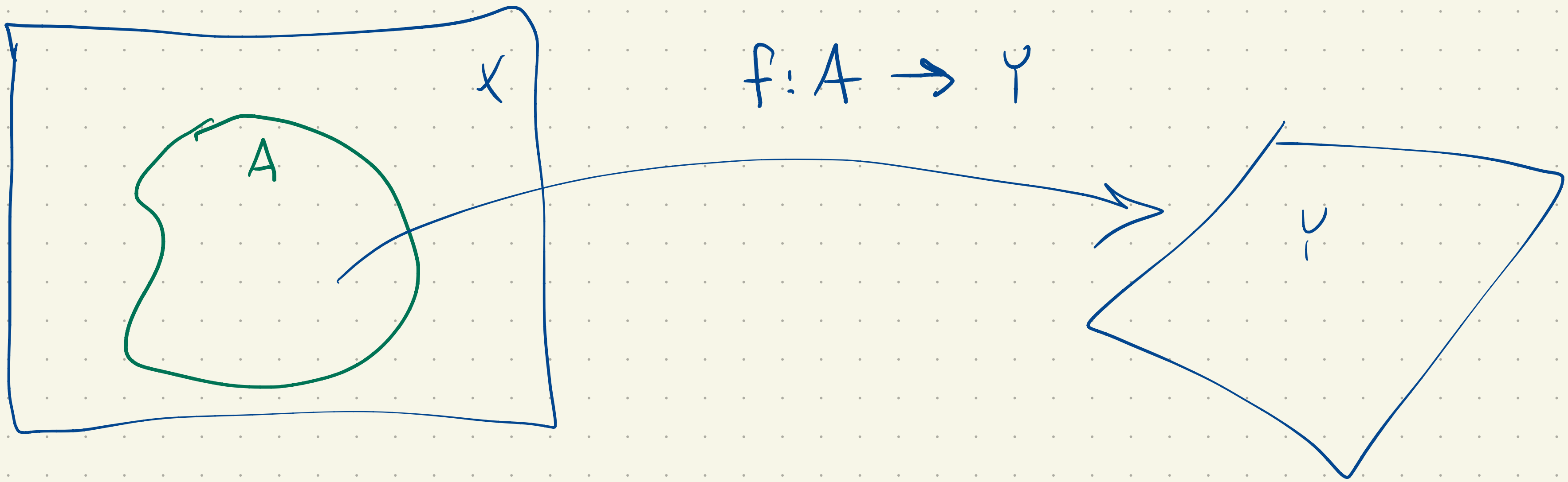
As $k \to \infty$, $d(a, a_{n_k}) \to 0$ and $1/n_k \to 0$.

Hence $d(a, b_{n_k}) \to 0$; i.e. $b_{n_k} \to a$ as well.

We then have, by continuity, $f(a_{n_k}) \to f(a)$ and $f(b_{n_k}) \to f(a)$.

But this is impossible since $d(f(a_{n_k}), f(b_{n_k})) \geq \varepsilon$ for all k.

$\square$



I'd like to extend f to all of $\bar{A}$.

continuous

$\bar{f} : \bar{A} \to Y$

$\bar{f}|_A = f$

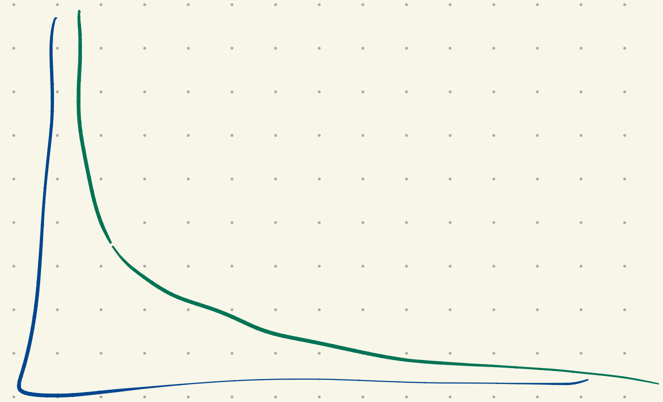
We call such an $\bar{f}$ a continuous extension.

$\bar{f}_1, \bar{f}_2$

$\bar{f}_1 = \bar{f}_2$ on A

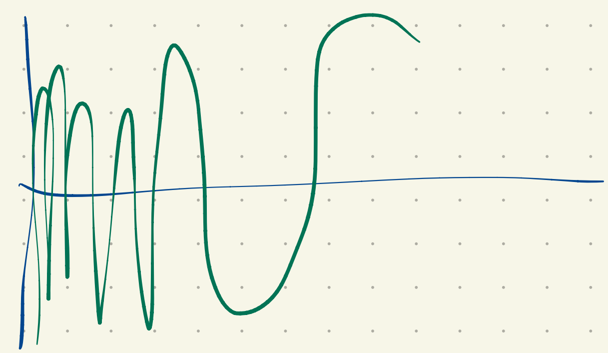
$f : (0,1] \to \mathbb{R}$

$f(x) = \frac{1}{x}$



$f : (0,1] \to \mathbb{R}$

$f(x) = \sin(1/x)$



Thm: Suppose $A \subseteq X$, $f: A \to Y$ is uniformly continuous, Y is complete, and $\overline{A} = X$. Then there exists a unique continuous function $\overline{f}: X \to Y$ such that $\overline{f}|_A = f$. Moreover, $\overline{f}$ is uniformly continuous.

Pf: Let $x \in X$ and let (a_n) be a sequence in A converging to x. Since (a_n) is Cauchy and since f is u.c., $(f(a_n))$ is also Cauchy.

Since Y is complete, $f(a_n) \to y$ for some $y \in Y$.

We define $\overline{f}(x) = y$.

[Is $\overline{f}$ well defined?]

Note that the value $\bar{f}(x)$ is independent of the choice of sequence. Indeed, if $z_n \to x$ then

$(a_1, z_1, a_2, z_2, \ldots)$ also converges to x and

by the argument above $(f(a_1), f(z_1), f(a_2), f(z_2), \ldots)$

converges to some limit $\hat{y}$. But this sequence has

a subsequence converging to y and hence $\hat{y} = y$.

But then $f(z_n) \to y$ as well.

I claim that $\bar{f}$ defined this way is uniformly continuous.

Indeed let $\varepsilon > 0$. Pick δ so if $a, b \in A$ and $d(a,b) < \delta$

then $d(f(a), f(b)) < \varepsilon/2$.

Now suppose $a, b \in X$ and $d(a,b) < \delta/3$.

Find sequences $(a_n), (b_n)$ in A with $a_n \to a$, $b_n \to b$.

Pick N so if $n \geq N$ $d(a_n, a) < \delta/3$ and $d(b_n, b) < \delta/3$.

Then if $n \geq N$

$$d(a_n, b_n) \leq d(a_n, a) + d(a,b) + d(b, b_n)$$
$$< \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3}.$$

So for $n \geq N$, $d(a_n, b_n) < \delta$ so $d(f(a_n), f(b_n)) < \frac{\varepsilon}{2}$.

Note: $d(\bar{f}(a), \bar{f}(b)) = \lim_{n \to \infty} d(f(a_n), f(b_n))$.

Hence $d(f(a), f(b)) \leq \varepsilon/2 < \varepsilon$. $\square$

[Note: $\bar{f}\big|_A = f$ using constant sequences]

$a \in A \qquad (a_n) \qquad a_n = a$

[$\bar{f}(a) = \lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(a) = f(a)$]