Given a compact metric space X, what subsets are compact?

$$A \subseteq X$$

$$\underbrace{\text{complete}}_{\Leftarrow \text{ closed}}, \quad \underbrace{\text{t.b.}}_{\text{free}}$$

Prop: If X is compact, $A \subseteq X$ is compact iff it is closed.

Warning: continuity preserves neither of completeness nor total boundedness

(see HW)

But: the magic combination of both is preserved.

Continuous functions map compact sets to compact sets.

Prop: If $f: X \to Y$ is continuous and $K \subseteq X$ is compact then $f(K)$ is as well.

Pf: Let (y_n) be a sequence in $f(K)$.

We can find a sequence (x_n) in K such that $y_n = f(x_n)$.

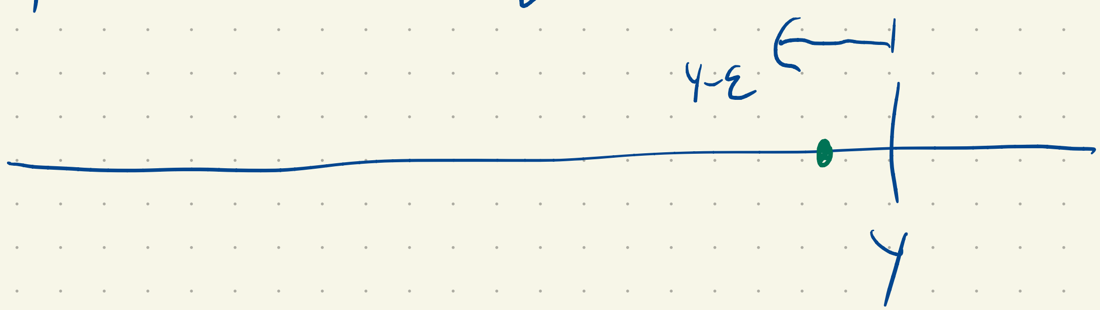
Since K is compact we can extract a convergent subsequence (x_{n_k}), converging to a limit $x \in K$.

By continuity $y_{n_k} = f(x_{n_k}) \to f(x) \in f(K)$. $\square$

Cor: If X is compact and nonempty and $f : X \to \mathbb{R}$ is continuous then f achieves a minimum and a maximum. That is, there exist $x_m, x_M \in K$ such that $f(x_m) \leq f(x) \leq f(x_M)$ for all $x \in X$.

Pf: Since X is compact, $f(X) \subseteq \mathbb{R}$ is compact also and therefore closed and bounded. Let $y = \sup(f(X))$; this exists since $f(X) \neq \emptyset$ and is bounded above.

Let (y_n) be a sequence in $f(X)$ converging to y.



Since $f(X)$ is closed $y \in f(X)$. So $y = f(x_M)$ for some x_M and $f(x_M) \geq f(x)$ for all $x \in X$.

X: compact

continuous

$C(X) = \{ f: X \to \mathbb{R} : f \text{ is cts} \}$

$$\|f\|_\infty = \sup \{ |f(x)| : x \in X \}$$

$$= \max \{ |f(x)| : x \in X \}$$ (well defined by above)

exercise: $\|\cdot\|_\infty$ is a norm on $C(X)$.

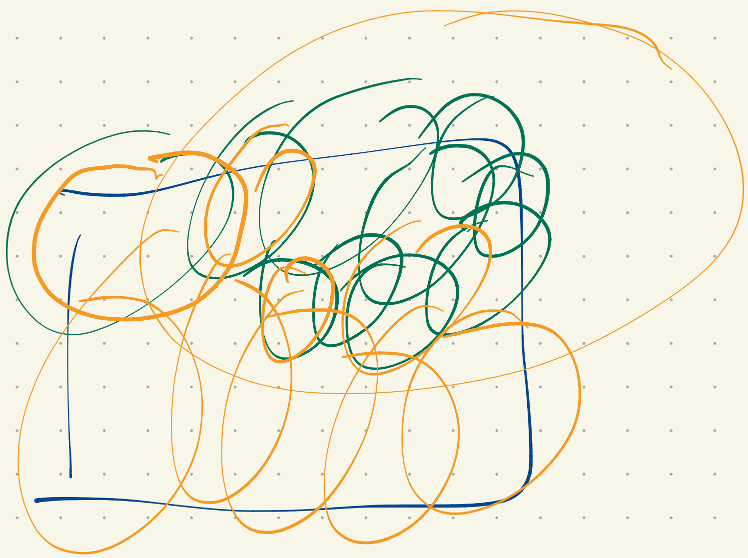
Def: A space $A \subseteq X$ is topologically compact if

whenever $\{U_\alpha\}_{\alpha \in I}$ is a collection of open

sets with $A \subseteq \bigcup_{\alpha \in I} U_\alpha$ → U_α's are an open cover

there exist

$U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}$ such that $\underbrace{A \subset \bigcup_{k=1}^{n} U_{\alpha_k}}_{\text{finite subcover}}$.



X not topologically compact:

There exists an open cover with no finite subcover

$\{U_\alpha\}$

$$X = \bigcup U_\alpha \Leftrightarrow \phi = \bigcap U_\alpha^c$$

red

$$X \neq \bigcup_{k=1}^{n} U_{\alpha_k} \Leftrightarrow \phi \neq \bigcap_{k=1}^{n} U_{\alpha_k}^c$$

salty

X is topologically compact if whenever $\{F_\alpha\}_{\alpha\in I}$ is a collection of closed sets such that $\bigcap_{k=1}^{n} F_{\alpha_k} \neq \phi$ for any finite subcollection, $\bigcap_{\alpha\in I} F_\alpha \neq \phi$.