Thm: Suppose X is complete. Then

$A \subseteq X$ is complete iff A is closed.

Pf: Suppose $A \subseteq X$ is complete. Suppose (a_n) is a sequence in A converges to limit $x \in X$. Since (a_n) is convergent it is Cauchy. Since A is complete (a_n) converges to a limit $a \in A$. But convergence in A implies convergence in X. Since limits are unique $a = x$.

Conversely suppose A is closed. Let (a_n) be Cauchy in A. Since X is complete and since (a_n) is also Cauchy in X, $a_n \to x$ for some $x \in X$, Since A is closed $x \in A$. So (a_n) converges in A to an element of A.

Def: A <u>Banach space</u> is a complete normed vector space.

$(C[0,1], L_2) \to$ not complete

E.g. $\mathbb{R}$, $(\mathbb{R}^n, \ell_2)$, ℓ_2

ℓ_p

[on HW: ℓ_1, ℓ_∞, c_0]

For normed vector spaces there is an alternative tool for demonstrating completeness.

$X \rightarrow$ normed vector space

$$\boxed{\sum_{k=1}^{\infty} x_k}$$

$x_k \in X$

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

$$S_n = \sum_{k=1}^{n} x_k$$

$S_n \rightarrow X \Rightarrow$

$$\sum_{k=1}^{\infty} x_k = X$$

A series in X is absolutely summable if $\sum_{k=1}^{\infty} \|x_k\|$ converges.

$$\underbrace{\sum_{k=1}^{\infty} x_k}_{X}$$

Convergent series need not be absolutely summable.

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \qquad \sum_{k=1}^{\infty} \frac{1}{k}$$

Recall: An absolutely convergent series of real numbers converges.

Thm: A normed ~~linear~~ vector space X is complete, ~~iff~~ every absolutely summable series in X converges.

Pf: Suppose X is complete. Let $\sum_{k=1}^{\infty} a_k$ be absolutely summable. Let $s_N = \sum_{k=1}^{N} a_k$.

Then if $N < M$

$$\|s_N - s_M\| = \left\| \sum_{k=N+1}^{M} a_k \right\| \leq \sum_{k=N+1}^{M} \|a_k\|. = t_M - t_N$$

$$= |t_M - t_N|$$

Let $t_N = \sum_{k=1}^{N} \|a_k\|$. Observe that

$$t_M - t_N = \sum_{k=N+1}^{M} \|a_k\|. \text{ Since the}$$

series is absolutely summable the sequence (t_N) is Cauchy as is the sequence (s_N).

Since X is complete, (s_N) converges, as does

$$\sum_{k=1}^{\infty} a_k.$$

Conversely, suppose absolutely summable series in X converge

Let (x_n) be a Cauchy sequence.

Find N_1 so if $n, m \geq N_1$, $\|x_n - x_m\| < \frac{1}{2}$.

Find $N_2 > N_1$ so if $n, m \geq N_2$ $\|x_n - x_m\| < \left(\frac{1}{2}\right)^2$

Continue inductively to build a sequence of indices

$$N_1 < N_2 < N_3 < \cdots$$

such that if $n, m \geq N_k$ $\|x_n - x_m\| < \left(\frac{1}{2}\right)^k$.

Consider the subsequence (x_{N_k}).

Observe

$$x_{N_k} = x_{N_1} + (x_{N_2} - x_{N_1}) + (x_{N_3} - x_{N_2}) + \cdots + (x_{N_k} - x_{N_{k-1}})$$

Note $\sum_{j=2}^{k} \|x_{N_j} - x_{N_{j-1}}\| \leq \sum_{j=2}^{k} \frac{1}{2^{j-1}} \leq 1$.

Hence $\sum_{j=2}^{k} (x_{N_j} - x_{N_{j-1}})$ is absolutely summable

and hence summable. Thus $\sum_{k=2}^{\infty} (x_{N_j} - x_{N_{j-1}})$ converges

as does (x_{N_k}). Thus (x_n) is a Cauchy

sequence with a convergent subsequence and converges.

HW: You will use this test to show l_1 is complete.

Def: A set $A \subseteq X$ is <u>compact</u> if every sequence in A has a convergent subsequence converging to a limit in A.

Lemma: Suppose $A \subseteq X$ is complete and totally bounded. Then A is compact.

Pf: Let (a_n) be a sequence in A.

Since A is totally bounded we can extract a Cauchy subsequence (a_{n_k}). Since A is complete,

(a_{n_k}) converges to a limit in A.

Lemma: Suppose $A \subseteq X$ is compact. Then it is totally bounded.

Pf: Let (a_n) be a sequence in A.

Since A is compact there exists a convergent and hence Cauchy subsequence.

Lemma: Suppose $A \subseteq X$ is compact. Then A is complete.

Pf: Let (a_n) be Cauchy in A. Since A is compact we can extract a convergent subsequence (a_{n_k}) converges to a limit in A. Since the original sequence is Cauchy, it also converges to the same limit in A.

$\boxed{\text{Thm: A set } A \subseteq X \text{ is compact iff it is complete and totally bounded.}}$

subsets of $\mathbb{R}$ closed and bounded

are compact $\Longleftrightarrow$