

Thm: Suppose X is complete. Then

$A \subseteq X$ is complete iff A is closed.

Pf: Suppose $A \subseteq X$ is complete. Suppose (a_n) is

a sequence in A converging to limit $x \in X$. Since

(a_n) is convergent it is Cauchy. Since A is complete

(a_n) converges to a limit $a \in A$. But convergence in A implies convergence in X . Since limits are unique $a = x$.

Conversely suppose A is closed. Let (a_n) be Cauchy

in A . Since X is complete and since (a_n) is also

Cauchy in X , $a_n \rightarrow x$ for some $x \in X$.

Since A is closed $x \in A$. So (c_n) converges in A to an element of A .

Def: A Banach space is a complete normed vector space.

$(C[0,1], L_2) \rightarrow$ not complete

E.g. \mathbb{R} , (\mathbb{R}^n, l_2) , l_2

\uparrow
 l_p

[on HW: l_1, l_∞, c_0]

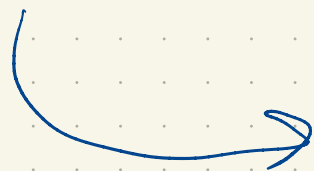
For normed vector spaces there is an alternative tool for demonstrating completeness.

$X \rightarrow$ normed vector space

$$\sum_{k=1}^{\infty} x_k$$

$$x_k \in X$$

$$\sum_{k=1}^{\infty} \frac{1}{k}$$



$$S_n = \sum_{k=1}^n x_k$$

$$S_n \rightarrow X \Rightarrow$$

$$\sum_{k=1}^{\infty} x_k = X$$

A series in X is absolutely summable

if $\sum_{k=1}^{\infty} \|x_k\|$ converges.

$$\underbrace{\sum_{k=1}^{\infty} x_k}_{X}$$

Convergent series need not be absolutely summable.

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

Recall: An absolutely convergent series \nearrow converges,
of real numbers.

Thm: A normed ~~linear~~ ^{vector} space X is complete, ~~iff~~
 every absolutely summable series in X converges.

Pf: Suppose X is complete. Let $\sum_{k=1}^{\infty} a_k$ be
 absolutely summable. Let $s_N = \sum_{k=1}^N a_k$.

Then $\forall N < M$

$$\|s_N - s_M\| = \left\| \sum_{k=N+1}^M a_k \right\| \leq \sum_{k=N+1}^M \|a_k\| = \epsilon_M - \epsilon_N = |\epsilon_M - \epsilon_N|$$

Let $\epsilon_N = \sum_{k=1}^N \|a_k\|$. Observe that

$$\epsilon_M - \epsilon_N = \sum_{k=N+1}^M \|a_k\|. \quad \text{Since the}$$

series is absolutely summable the sequence (t_n) is Cauchy as is the sequence (s_n) .

Since X is complete, (s_n) converges, as does

$$\sum_{k=1}^{\infty} a_k.$$

Conversely, suppose absolutely summable series in X converge

Let (x_n) be a Cauchy sequence.

Find N_1 so if $n, m \geq N_1$, $\|x_n - x_m\| < \frac{1}{2}$.

Find $N_2 > N_1$ so if $n, m \geq N_2$, $\|x_n - x_m\| < \left(\frac{1}{2}\right)^2$

Continue inductively to build a sequence of indices

$$N_1 < N_2 < N_3 < \dots$$

such that if $n, m \geq N_k$ $\|x_n - x_m\| < \left(\frac{1}{2}\right)^k$.

Consider the subsequence (x_{N_k}) .

Observe

$$x_{N_k} = x_{N_1} + (x_{N_2} - x_{N_1}) + (x_{N_3} - x_{N_2}) + \dots + (x_{N_k} - x_{N_{k-1}})$$

Note $\sum_{j=2}^k \|x_{N_j} - x_{N_{j-1}}\| \leq \sum_{j=2}^k \frac{1}{2^{j-1}} \leq 1$.

Hence $\sum_{j=2}^k (x_{N_j} - x_{N_{j-1}})$ is absolutely summable

and hence summable. Thus $\sum_{k=2}^{\infty} (x_{N_j} - x_{N_{j-1}})$ converges

as does (x_{N_k}) . Thus (x_n) is a Cauchy

sequence with a convergent subsequence and converges.

HW: You will use this test to show \mathbb{Q} is complete.

Def: A set $A \subseteq X$ is compact if every sequence in A has a convergent subsequence converging to a limit in A .

Lemma: Suppose $A \subseteq X$ is complete and totally bounded. Then A is compact.

Pf: Let (a_n) be a sequence in A .

Since A is totally bounded we can extract a Cauchy subsequence (a_{n_k}) . Since A is complete,

(a_n) converges to a limit in A ,

Lemma: Suppose $A \subseteq X$ is compact. Then it is totally bounded.

Pf: Let (a_n) be a sequence in A .

Since A is compact there exists a convergent and hence Cauchy subsequence.

Lemma: Suppose $A \subseteq X$ is compact. Then A is complete.

Pf: Let (a_n) be Cauchy in A . Since A is compact we can extract a convergent subsequence (a_{n_k}) converging to a limit in A . Since the original sequence is Cauchy, it also converges to the same limit in A .

Thm: A set $A \subseteq X$ is compact iff it is complete and totally bounded.

subsets of \mathbb{R} closed and bounded
are compact \Leftrightarrow