So $d(x_{n_k}, x_{n_j}) \leq \operatorname{diam}(A_K) \leq \frac{1}{K} < \varepsilon$.

$\square$

Thm: A set $A \subseteq X$ is totally bounded iff every sequence in A has a Cauchy subsequence.

Pf: Suppose A is totally bounded. Consider a sequence (x_n) in A. Since $\{x_n : n \in \mathbb{N}\} \subseteq A$, it is totally bounded and the previous lemma shows it admits a Cauchy subsequence.

Conversely suppose A is not totally bounded.

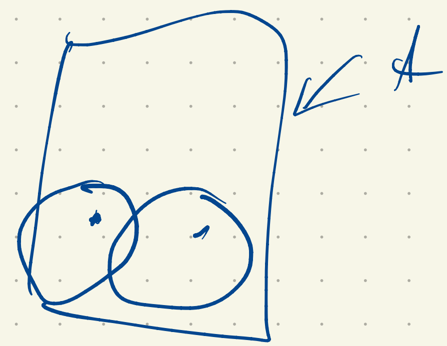
[Job: show there exists a sequence with no Cauchy subseq.]

Then there exist $\varepsilon_0 > 0$ such there does not exist an ε_0-net.

$B_\varepsilon(x_1)$

Pick $x_1 \in A$. Since $\{x_1\}$ is not an ε-net $\{x_1, x_2\}$ we can find $x_2 \in A$, $x_2 \notin B_\varepsilon(x_1)$. Since $\{x_1, x_2\}$ IS not an ε_0-net we can find $x_3 \in A$ $x_3 \notin \bigcup_{i=1}^{2} B_{\varepsilon_0}(x_i)$.

Continuing inductively we can construct a sequence (x_n) such that if $n \neq m$



$$d(x_n, x_m) \geq \varepsilon_0$$

This sequence has no Cauchy subsequence. $\square$

Cor: Bolzano-Weierstrass

Every bounded sequence of real numbers has a convergent subsequence.

Pf: Suppose $(x_n) \subseteq [-R, R]$ for some $R > 0$.

Then $A = \{x_n : n \in \mathbb{N}\}$ is totally bounded as $[-R, R]$ is and hence it admits a Cauchy subsequence. Cauchy sequence of real numbers converge. $\square$.

1-2 punch: 1) use total boundedness to extract a Cauchy subsequence

2) Use completeness to verify convergence.

Def: A space X is <u>complete</u> if every Cauchy sequence in X converges.

e.g. ① $\mathbb{R}$

$$d_1((a,b),(c,d)) = |a-c| + |b-d|$$

② $\mathbb{R}^2$ with ℓ_1 norm.

Pf: Suppose $(z_n) = ((x_n, y_n))$ is Cauchy.

Observe $|x_n - x_m| \leq \| z_n - z_m \|_1$.

Hence (x_n) is Cauchy and converges to some limit x.

Similarly (y_n) converges to a limit y.

We claim $z_n \to (x,y)$. Indeed

$$\| z_n - (x,y) \|_1 = | x_n - x | + | y_n - y | \to 0.$$

Notice: again two steps

a) Exhibit a candidate (x,y).

b) $z_n \to (x,y)$

Let's show ℓ_2 is complete.

Suppose (x_n) is a Cauchy sequence in ℓ_2.

$x_n \in \ell_2$ $\quad x_n(k) \in \mathbb{R}$ $\quad x_n = (x_n(1), x_n(2), x_n(3), \ldots \ldots)$

We need a candidate.

$$|x_n(1) - x_m(1)|^2 \leq \sum_{k=1}^{\infty} |x_n(k) - x_m(k)|^2 = \|x_n - x_m\|_2^2$$

Hence $(x_n(1))$ is Cauchy in $\mathbb{R}$ and converges to some limit $y(1)$.

Proceeding similarly, each $(x_n(k))$ converges to a limit $y(k)$. Let $y = (y(1), y(2), y(3), \ldots)$.

a) Is $y \in \ell_2$?

b) Does $x_n \to y$ in ℓ_2?

$x_n(k) \to y(k)$

$|x_n(k)|^2 \to |y(k)|^2$

a) Observe for each K

$$\sum_{k=1}^{K} |y(k)|^2 = \lim_{n\to\infty} \sum_{k=1}^{K} |x_n(k)|^2$$

$$= \limsup_{n\to\infty} \sum_{k=1}^{K} |x_n(k)|^2$$

$$\leq \limsup_{n\to\infty} \|x_n\|_2^2$$

Since (x_n) is Cauchy, it is bounded and thus exists M such that $\|x_n\|_2 \le M \quad \forall n$.

So for each K, $\sum_{k=1}^{K} |y(k)|^2 \le M^2$.

Thus $\|y\|_2 \le M$. So $y \in \ell_2$.

$\sum_{n=1}^{\infty} a_n \qquad a_n \geq 0$

$s_m = \sum_{n=1}^{m} a_n$

$s_m \le M$

$\sum_{n=1}^{\infty} a_n \le M$.

---

b_n, increasing $\qquad b_n \le M \qquad b_1 \le b_n \le M$

$b_n \to b$

Does $x_n \to y$?

Let $\varepsilon > 0$. Since (x_n) is Cauchy we can find

N so if $n, m \geq N$, $\|x_n - x_m\|_2 < \varepsilon$.

Observe for each K, if $n \geq N$

$$\sum_{k=1}^{K} |y(k) - x_n(k)|^2 = \lim_{m \to \infty} \sum_{k=1}^{K} |x_m(k) - x_n(k)|^2$$

$$\leq \limsup_{m \to \infty} \|x_m - x_n\|_2^2$$

$$\leq \varepsilon^2.$$

But then if $n \geq N$, $\|y - x_n\|_2 \leq \varepsilon$.

Thus $x_n \to y$.

HW: ℓ_1, ℓ_∞, c_0 are all complete

$(C[0,1], L_\infty)$: also complete (IOU)

$(C[0,1], L_1)$: not complete