an isometry.

Exercise: Isometries are continuous,

$$\varepsilon = \delta$$

Key Property of $\mathbb{R}$: Bounded sequences have convergent subsequences.

This is false, in general, for metric spaces.

Two issues.

$\mathbb{Q}$: $3, 3.1, 3.14, 3.141, \ldots$

n'th position

ℓ_∞ $e_n = (0, 0, \ldots, 0, 1, 0, \ldots)$

(e_n) is bounded in ℓ_∞ $\quad (0, \dots, 0, \underset{\uparrow n}{1}, 0, \dots, 0, \overset{\swarrow m}{-1}, 0, \dots)$

$e_n - e_m$

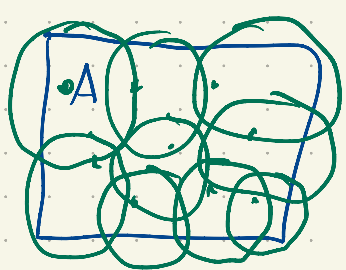
$$\|e_n - e_m\|_\infty = 1 \quad \text{if} \quad n \neq m$$

So no Cauchy subsequence.

Def: A set $A \subseteq X$ is <u>totally bounded</u> if for all $\varepsilon > 0$ there are finitely many points $x_1, \dots, x_n \in X$ such that

$$A \subseteq \bigcup_{i=1}^{n} B_\varepsilon(x_i).$$

Such a collection of points is called an <u>ε-net</u> for A.



$\varepsilon = 1$

Exercise: A totally bounded set is bounded.

[is the converse true? No, but yes for $\mathbb{R}$]

$$A = \{e_n\} \subseteq l_\infty$$

Bounded but not totally bounded.

There is no $1/2$-net.

$$z_1, \dots, z_k \in l_\infty$$

$$A \nsubseteq \underbrace{\bigcup_{j=1}^{k} B_{1/2}(z_j)}$$

$\underbrace{B_{1/2}(z_j)}$

↳ can contain at most one e_n

↳ contains at most k elements of A.

Alternative Characterization

Lemma: A set A is totally bounded iff for all $\varepsilon > 0$ there exist $A_1, \ldots, A_n \subseteq A$ such that $\operatorname{diam} A_k < \varepsilon$ for all k and $A \subseteq \bigcup_{k=1}^{n} A_k$.

Pf: Suppose A is totally bounded. Let $\varepsilon > 0$ and consider an $\varepsilon/4$-net $\{x_1, \ldots, x_n\}$. Let $A_k = B_{\varepsilon/4}(x_k) \cap A$. Note: $\operatorname{diam} A_k \leq \frac{\varepsilon}{2} < \varepsilon$.

Moreover $A \subseteq \bigcup_{k=1}^{n} B_{\varepsilon/2}(x_k)$ and hence

$$A = A \cap A \subseteq \bigcup_{k=1}^{n} A \cap B_{\varepsilon/2}(x_k) = \bigcup_{k=1}^{n} A_k.$$

Conversely, suppose $A_1, \ldots, A_n$ are subsets of

A with diameter less than ε for some $\varepsilon > 0$, with $A \subseteq \bigcup A_k$.

WLOG each $A_k \neq \emptyset$

For each k pick $x_k \in A_k$. Since $\operatorname{diam}(A_k) < \varepsilon$,

$B_\varepsilon(x_k) \supseteq A_k$.

$d(x,y) \leq \operatorname{diam}(A_k)$
$x, y \in A_k$
$d(x_k, y) \leq \operatorname{diam}(A_k)$
$B_{\operatorname{diam}(A_k)}(x_k) \supseteq A_k$

Then $\bigcup_{k=1}^{n} B_\varepsilon(x_k) \supseteq \bigcup_{k=1}^{n} A_k \supseteq A$.

So $\{x_1, \dots, x_n\}$ is an ε-net.

$\square$

Cor: $[0,1]$ is totally bounded. $I_k = \left[\frac{k-1}{n}, \frac{k}{n}\right]$ $k = 1, \dots, n$.

$\operatorname{diam}(I_k) = \frac{1}{n}$ $\bigcup_{k=1}^{n} I_k = [0,1]$

Exercise $[-R, R]$ is t.b. $\forall R > 0$.

Exercise If A is t.b. and $B \subseteq A$ then B is t.b.

Exercise: Bounded subsets of $\mathbb{R}$ are t.b.

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total boundedness has a lot to do with Cauchy sequences.

Lemma: Suppose (x_n) is Cauchy in X. Then

$$\{x_n : n \in \mathbb{N}\}$$ is totally bounded.

Pf: Let $\varepsilon > 0$, [Job: exhibit an ε-net]. Since the sequence is Cauchy there exists N so if $n, m \geq N$ $d(x_n, x_m) < \varepsilon$. We claim $\{x_1, x_2, \ldots, x_N\}$ is an ε-net.

$B_\varepsilon(x_j)$ $\quad 1 \le j \le N$ $\quad x_k$

$k \ge N$

$d(x_k, x_N) < \varepsilon$

$x_k \in B_\varepsilon(x_N)$

Indeed if $n \ge N$, $x_n \in B_\varepsilon(x_N)$ and if $n < N$, $x_n \in B_\varepsilon(x_n)$.

$\square$

Lemma: Given a sequence (x_n), if $\{x_n : n \in \mathbb{N}\}$ is totally bounded, then the sequence admits a Cauchy subsequence.

Pf: If $\{x_n : n \in \mathbb{N}\}$ is finite we can extract a constant subsequence. Otherwise let $A_0 = \{x_k : k \in \mathbb{N}\}$.

Since A_0 is totally bounded there exists a subset A_1 with $\operatorname{diam} A_1 \leq 1$ such that A_1 contains infinitely many terms.

Since A_1 is totally bounded there exists $A_2 \subseteq A_1$ with $\operatorname{diam} A_2 \leq \frac{1}{2}$ and A_2 contains infinitely many terms of the sequence.

Continuing inductively we can find subsets

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$$

where $\operatorname{diam} A_k \leq 1/k$ and each A_k contains infinitely many terms of the sequence.

We extract a subsequence as follows.

Pick n_1 such that $x_{n_1} \in A_1$.

Pick n_2 such that $n_2 > n_1$ and $x_{n_2} \in A_2$.

This is possible since $x_1, x_2, \dots, x_{n_1}$ does not exhaust the infinite set A_2.

Continuing inductively we construct a subsequence x_{n_k} where each $x_{n_k} \in A_k$.

To see that the sequence is Cauchy, let $\epsilon > 0$.

[Job: show there $\exists K$] Pick $K \in \mathbb{N}$ so that $1/K < \varepsilon$.

Suppose $k, j \geq K$. Then $x_{n_k} \in A_k \subseteq A_K$. Similarly, $x_{n_j} \in A_K$.

So $d(x_{n_k}, x_{n_j}) \leq \operatorname{diam}(A_k) \leq \frac{1}{k} < \varepsilon.$

$\square$

Thm: A set $A \subseteq X$ is totally bounded iff every sequence in A has a Cauchy subsequence.