$F : C[0,1] \to \mathbb{R}$

$$F(f) = f(0)$$

$F(\exp) = \exp(0) = 1$

$F(\sin) = 0$

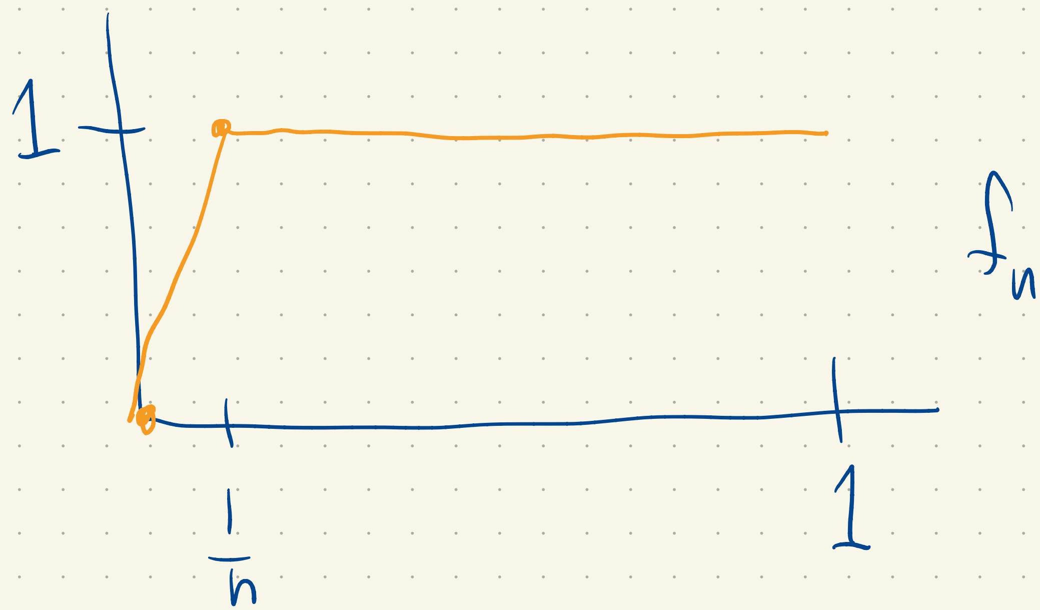
If $C[0,1]$ is given the L^p norm is F continuous?

$$\left[\int_0^1 |f(x)|^p \, dx \right]^{1/p}$$

$p = 1$

$$\int_0^1 |f(x)| \, dx$$

$$F(f) = f(0)$$



$f_n \to 1$

$F(f_n) = 0$

$F(1) = 1$

$f_n \longrightarrow 1$

$F(f_n) \not\to F(1)$

This is not a continuous function.

$(C[0,1], L_\infty)$ F is continuous!

$$\|f\|_\infty = \sup_{x \in [0,1]} |f(x)|$$

Suppose $f_n \to f$ in L_∞.

$\forall n$ $$|f_n(0) - f(0)| \leq \| f_n - f \|_\infty$$

$$|F(f_n) - F(f)| \leq \|f_n - f\|_\infty$$

If $f_n \to f$ in L_∞, $|F(f_n) - F(f)| \to 0$

$F(f_n) \to F(f)$.

$$G(f) = \int_0^1 f(x)\,dx \qquad G : C[0,1] \to \mathbb{R}$$

Is G cts w.r.t L_1 norm? $\qquad f, g \in C[0,1]$

$$|G(f) - G(g)| = \left| \int_0^1 f(x) - g(x)\,dx \right|$$

$$\leq \int_0^1 |f(x) - g(x)|\,dx$$

$$= \|f - g\|_1$$

$f_n \to f \qquad |G(f_n) - G(f)| \leq \|f_n - f\|_1$

$\Rightarrow \quad G(f_n) \to G(f).$

So, yes!

$(C[0,1], L_1) \to (C[0,1], L_\infty)$

What about w.r.t. L_∞?

$f_n \to f$ in L_1

$f_n \not\to f$ in L_∞

Exercise: Show $(C[0,1], L_\infty) \longrightarrow (C[0,1], L_1)$ $\quad \|f\|_1 \le \|f\|_\infty$

$$f \longmapsto f$$

is continuous.

Exercise: If $f: X \to Y$ and $g: Y \to Z$ are continuous then $g \circ f$ is continuous.

Exercise: G defined above is cts. w.r.t. L_∞ norm.

[Do no work!]

$$\|f\|_1 = \int_0^1 |f(x)|\,dx \leq \int_0^1 \|f\|_\infty\,dx \leq \|f\|_\infty$$

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

$x_n \to x$ $\quad$ $f(x_n) \to f(x)$ $\quad$ $g(f(x_n)) \to g(f(x))$

$P[0,1] \subseteq C[0,1]$

polynomials

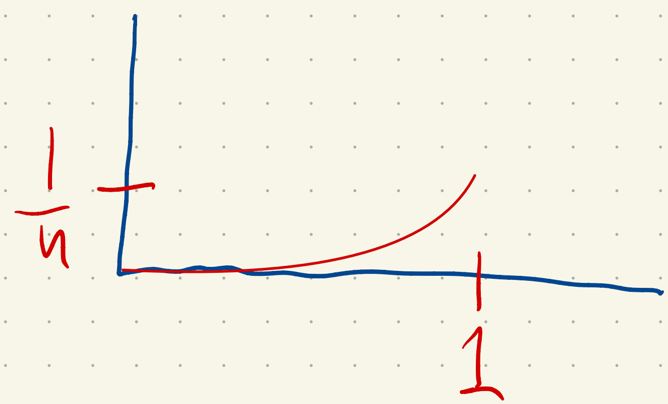
$D: P[0,1] \to P[0,1]$

$p \longmapsto p'$

is this continuous (w.r.t. L_∞ norm)

$f_n(x) = \frac{1}{n} x^n$

$f_n \longrightarrow 0$



$$\|f_n\|_\infty = \frac{1}{n}$$

$$\|f_n - 0\|_\infty \to 0$$

$$f_n'(x) = x^{n-1}$$

If D were continuous

then $D(f_n) \to D(0)$

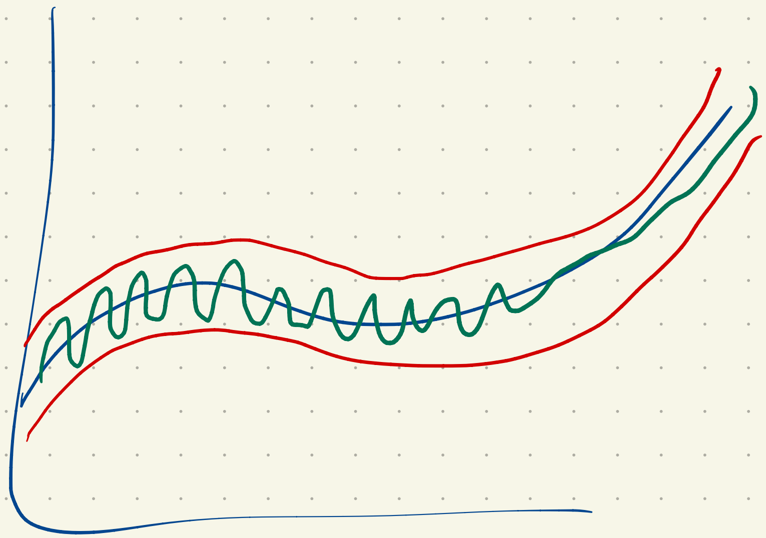
$$f_n \to 0$$

$$D(f_n) = x^{n-1}$$

i.e. $D(f_n) \to 0$

$$\|D(f_n)\|_\infty = 1$$

$$\|D(f_n)\|_\infty \to \|0\|_\infty$$



Suppose $f: X \to Y$ is continuous and is a bijection.

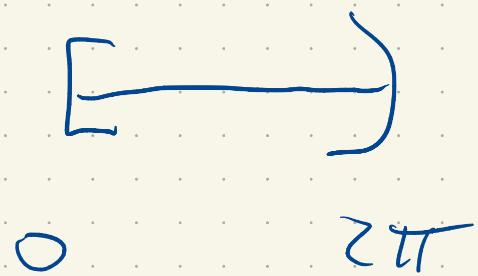
Is f^{-1} necessarily continuous?

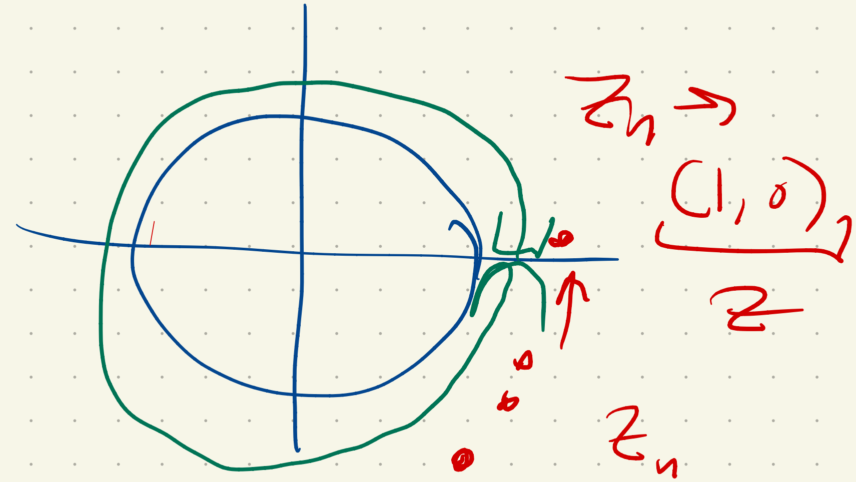
No.

Euclidean.

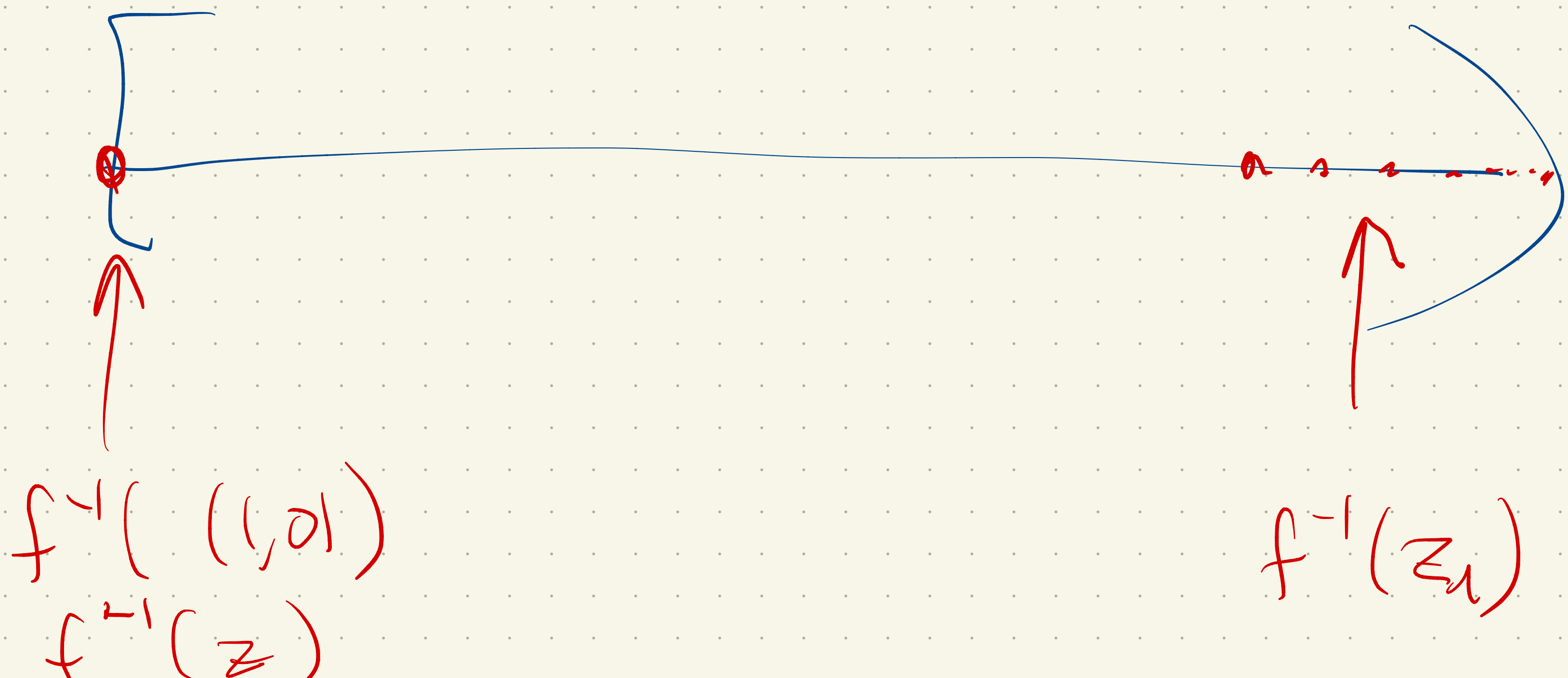
$$f: [0, 2\pi) \longrightarrow S^1 = \{ z \in \mathbb{R}^2 : |z| = 1 \}$$

$$f(\theta) = (\cos(\theta), \sin(\theta))$$





$z_n \to z$ but $f^{-1}(z_n) \not\to f^{-1}(z)$



Def: A function $f: X \to Y$ is an <u>isometry</u>

if $\forall \; x_1, x_2 \in X \quad d(x_1, x_2) = d(f(x_1), f(x_2))$

(distance preserving maps)

e.g. $f: \mathbb{R} \to \mathbb{R}$

$f(x) = -x + 18$

Exercise: Show that an isometry $\mathbb{R} \to \mathbb{R}$ is uniquely determined by its action on two points.

$$a \longmapsto \alpha \qquad a \neq b$$
$$b \longmapsto \beta$$

Use this to show that every isometry $\mathbb{R} \to \mathbb{R}$ has the form $\pm x + c$

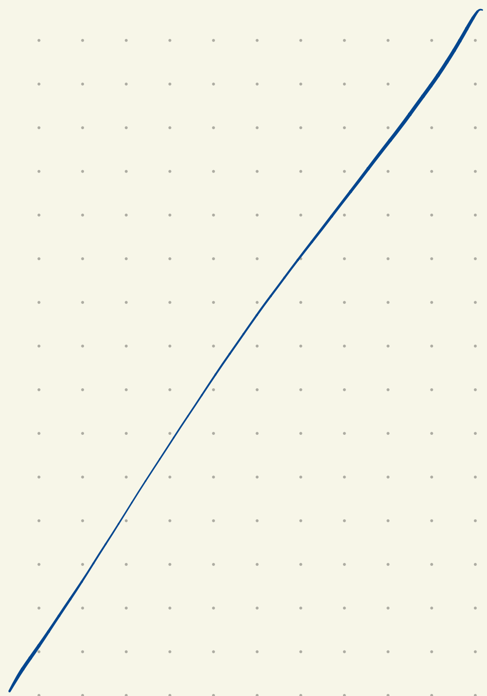
Exercise: Isometries are always injective.

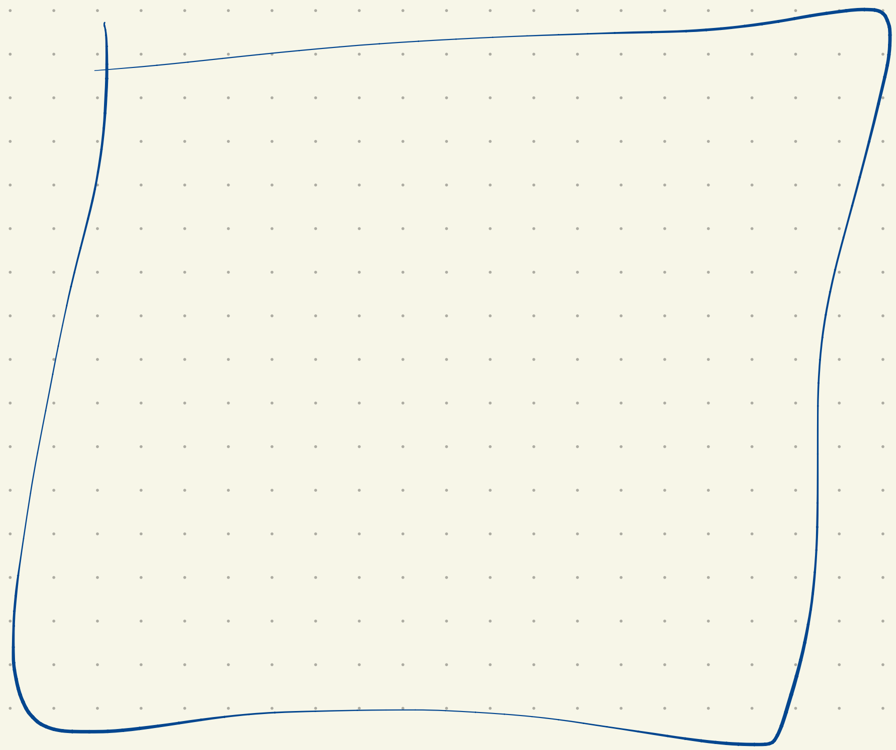
Suppose $f : X \to Y$ is an isometry.

Suppose $a, b \in X$ and $f(a) = f(b)$. [Job: show $a = b$]

$$d(f(a), f(b)) = 0 \qquad d(a,b) = 0$$
$$a = b$$

Are isometries always surjective?





$$x \longrightarrow (x, 0)$$

Exercise: The inverse of a surjective isometry is an isometry.

Exercise: Isometries are continuous.

$$\varepsilon = \delta$$