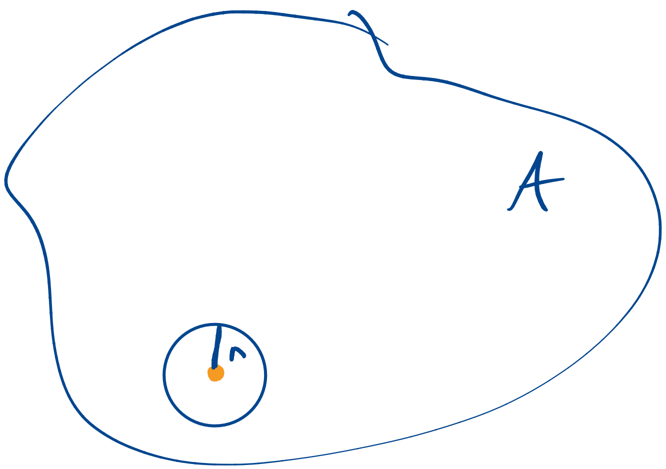
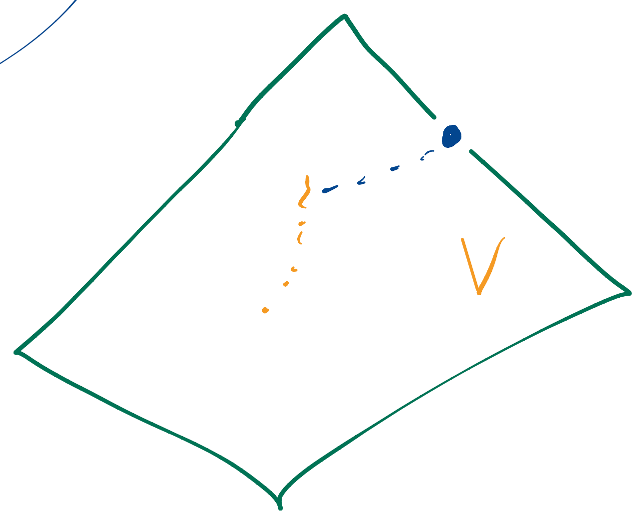
Exercise: A finite intersection of open sets is open

A finite union of closed sets is closed.



X

$\forall x \in A \; \exists \, r > 0$ with $B_r(x) \subseteq A$



Def: Given a set $A \subseteq X$, $\bar{A}$ (the closure of A) is the intersection of all closed sets containing A.

Note: X is a closed set containing A

$\bar{A}$ is closed and is the smallest closed set containing A.

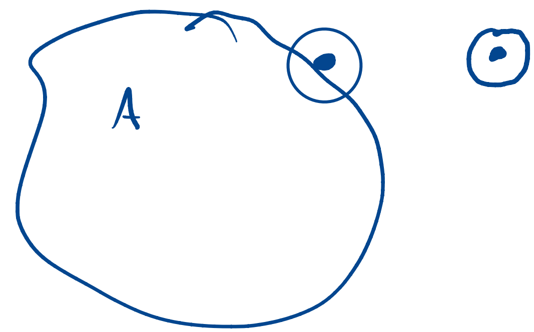
Prop: Let $A \subseteq X$ and let $x \in X$. TFAE

1) $x \in \bar{A}$
2) $\forall \epsilon > 0 \quad B_\epsilon(x) \cap A \neq \phi \quad \left(\exists y \in A,\ d(x,y) < \epsilon \right)$
3) $\exists$ a sequence in A converging to x.

Pf: 1) $\Rightarrow$ 2) via $!2) \Rightarrow !1)$

Suppose for some $\varepsilon > 0$

$B_\varepsilon(x) \cap A = \phi$.

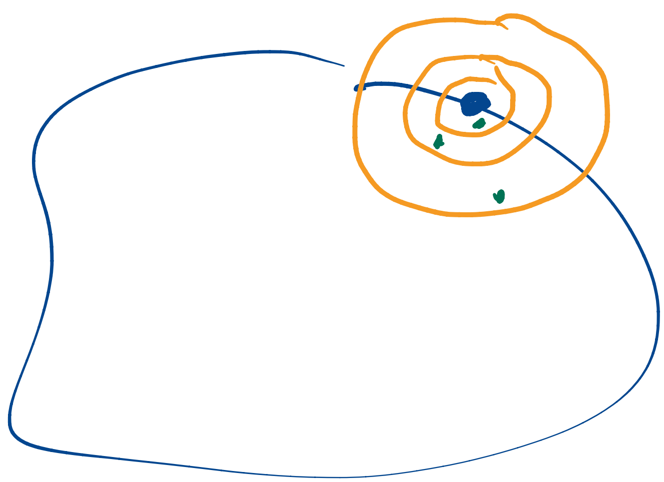


Then $[B_\varepsilon(x)]^c$ is a closed set

that contains A and hence also contains $\bar{A}$.

Since $x \in B_\varepsilon(x)$, $x \notin \bar{A}$.

2) $\Rightarrow$ 3)



(We did a proof just like this last class; use $\varepsilon = \frac{1}{n}$)

3) $\Rightarrow$ 1)

Suppose (x_n) is a sequence in A converges to x.

Then (x_n) is also a sequence in the closed set $\overline{A}$.

By the sequential characterization of closed sets, $x \in \overline{A}$.

$\overline{\mathbb{Q}} = \mathbb{R}$ $\qquad\qquad$ $[0, 1]$

$x \in \mathbb{R}$ $\qquad$ $q_n \to x$ $\qquad$ $q_n \in \mathbb{Q}$

[use decimal expansions]

$\overline{A}$ is the set of points in X that can be approximated as well as you want by points in A.

Def: We say a set A is <u>dense</u> in X if $\overline{A} = X$.

A space X is <u>separable</u> if it admits a countable dense set.

Countable is manageable. Separable is almost as manageable.

$P[0,1] \subseteq C[0,1]$

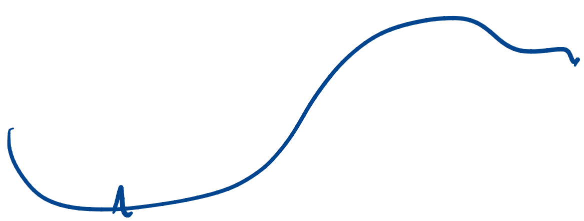
$\downarrow$

poly'ls restricted to $[0,1]$

Is $P[0,1]$ open? closed? dense?

$$\overline{P[0,1]} = C[0,1]$$

we'll prove this!



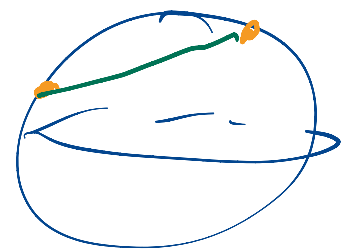
Indeed polynomials with rational coefficients are dense and hence $C[0,1]$ is separable.

Metrics on related spaces

If $A \subseteq X$ and X is a metric space, so is A

in its own right.

$$d_A(x,y) = d_X(x,y)$$



Exercise: $U \subseteq A$ is open $\iff \exists\, V$, open in X, $V \cap A = U$

$W \subseteq A$ is closed $\iff \exists\, Z \subseteq X$, closed in X, $Z \cap A = W$

Product spaces: X, Y metric spaces

$$X \times Y = \{(x,y) : x \in X,\ y \in Y\}$$

$d_{X \times Y}$ want $(x_n, y_n) \to (x, y)$ ~~iff~~

$x_n \to x$ and $y_n \to y$

$$d_{X\times Y}\big((x_0,y_0),(x_1,y_1)\big) = \begin{cases} d_X(x_0,x_1) + d_Y(y_0,y_1) \\ \left(d_X(x_0,x_1)^2 + d_Y(y_0,y_1)^2\right)^{1/2} \\ \max\big(d_X(x_0,x_1), d_Y(y_0,y_1)\big) \end{cases}$$

You'll see this on HW.

These metrics all determine the same convergent sequences and hence the same closed sets and hence the same open sets.

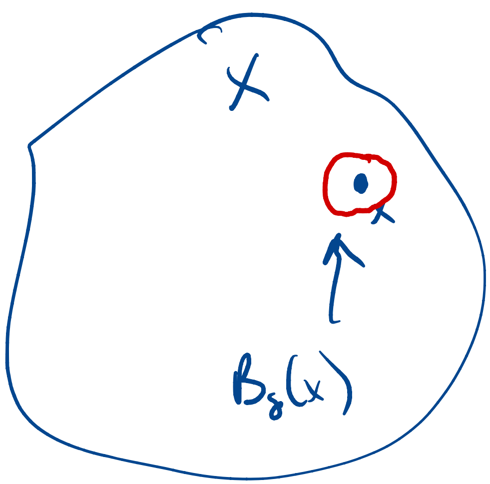
Continuity:

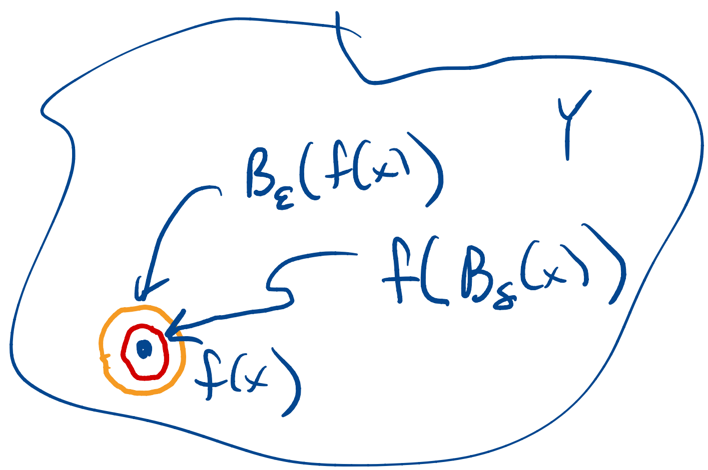
Def: We say $f: X \to Y$ is continuous at $x \in X$ if for all $\varepsilon > 0$ there exists $\delta > 0$ so that if $z \in B_\delta(x)$ $f(z) \in B_\varepsilon(f(x))$.

$B_r(x) = \{y \in X : d(x,y) < r\}$

$$\left[f(B_\delta(x)) \subseteq B_\varepsilon(f(x)) \right]$$

$$f(A) = \{f(a) : a \in A\}$$





Def: A function $f: X \to Y$ is <u>sequentially continuous</u> at $x \in X$ if whenever $x_n \to x$ in X, $f(x_n) \to f(x)$ in Y.

$\exists$ $x_n \to x$ in X such that $f(x_n) \not\to f(x)$

Prop: A function is continuous at x if and only if it is sequentially continuous at x.

Pf: Suppose f is continuous at x and $x_n \to x$.

[Job: $f(x_n) \to f(x)$]

Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$. Since $x_n \to x$ there exists N so if $n \geq N$ then $x_n \in B_\delta(x)$. Hence if $n \geq N$ $f(x_n) \in B_\varepsilon(f(x))$ and hence $f(x_n) \to f(x)$.

Conversely suppose f is not continuous at x.

Since f is not continuous, there exists $\varepsilon > 0$ such that for all $\delta > 0$ $f(B_\delta(x)) \not\subseteq B_\varepsilon(f(x))$. So for each $n \in \mathbb{N}$ we can pick $x_n \in B_{1/n}(x)$ with

$$d(f(x_n), f(x)) \geq \varepsilon.$$

But then $x_n \to x$ and $f(x_n) \not\to f(x)$.