Exercise: Show that there does not exist some $g \in C[0,1]$ such that $g_n \longrightarrow g$ (w.r.t. the L_1 norm).

Suppose $g_n \longrightarrow g \in C[0,1]$.

Let $1 > x_0 > \frac{1}{2}$. Then $g_n = 1$ on $[x_0, 1]$ for n sufficiently large.

Then

for n large enough

$$\int_{x_0}^{1} |g(x) - 1| \, dx = \int_{x_0}^{1} |g(x) - g_n(x)| \, dx \leq \|g - g_n\|_1 \to 0.$$

Hence $\int_{x_0}^{1} |g(x) - 1| \, dx = 0$. Hence $g(x) = 1$ on $[x_0, 1]$ for all $x_0 > \frac{1}{2}$. Similarly, $g(x) = 0$ on $[0, 1/2]$.

Exercise: if $f(x) \geq 0$ on $[a,b]$ and is continuous

and $\int_a^b f(x)\,dx = 0$ then $f(x) = 0 \ \forall x \in [a,b]$.

$$g(x) = \begin{cases} 0 & 0 \leq x \leq 1/2 \\ 1 & \frac{1}{2} < x \leq 1 \end{cases}$$ and there is no such

element in $C[0,1]$.

Def: Let X be a metric space.

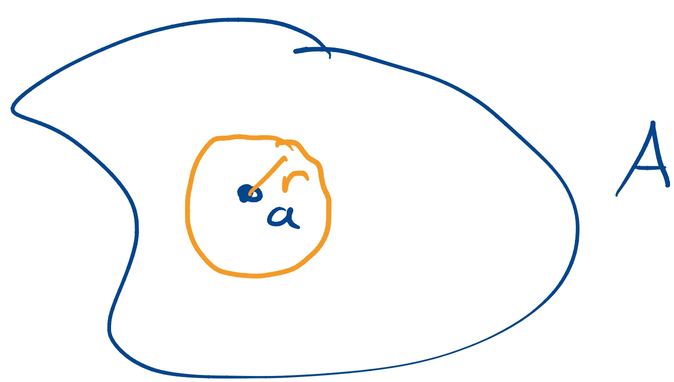
Given $x \in X$ and $r > 0$, $B_r(x) = \{y \in X : d(x,y) < r\}$

$$\overline{B_r}(x) = \{x \in X : d(x,y) \le r\}.$$

closed - — - radius r.

open ball of radius r

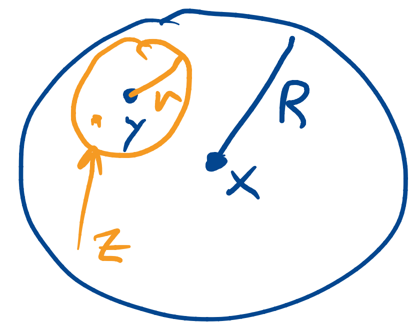
Def: A set $A \subseteq X$ is <u>open</u> if for all $a \in A$ there exists $r > 0$ with $B_r(a) \subseteq A$.



Examples: $(a,b) \subseteq \mathbb{R}$

$\phi \subseteq \mathbb{R}$

$B_R(x) \subseteq X$



Let $y \in B_R(x)$. Let $r = R - d(x,y)$, so $r > 0$.

$[d(x,y) < R]$

Then if $z \in B_r(y)$

$$\begin{aligned} d(z,x) &\leq d(z,y) + d(y,x) \\ &< r + d(y,x) \\ &= R. \end{aligned}$$

So $B_r(y) \subseteq B_R(x)$.

$$r = R - d(x,y)$$

$$d(x,y) = R - r$$

$$r + d(y,x) = r + R - r$$

$$= R$$

$$A = \{ f \in C[0,1] : f(x) > 0 \ \forall x \in [0,1] \}$$

Is A open in $C[0,1]$?

$(C[0,1], L_1)$?

Yes for $(C[0,1], L_\infty)$. (Exercise:

If $f \in C[0,1]$ and $f(x) > 0 \ \forall x$ let

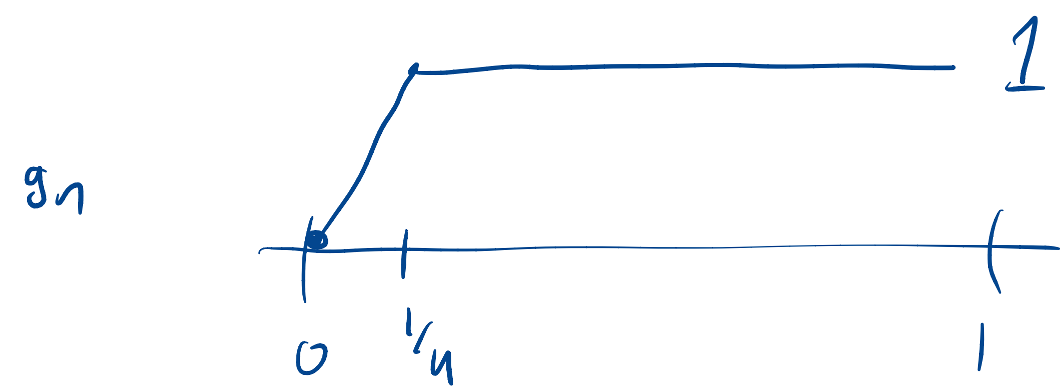
$m = \min f > 0$.

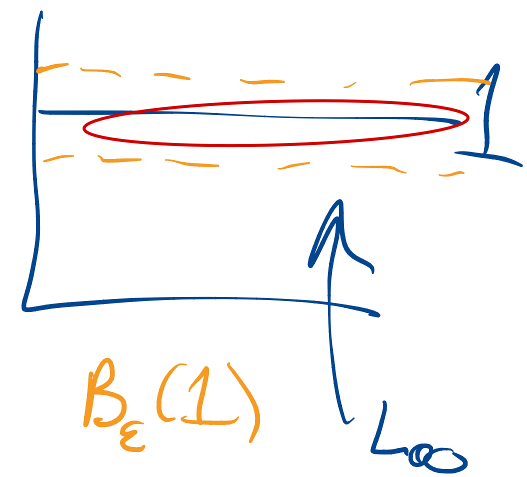
Then $B_{1/m}(f) \subseteq A$.

But for $(C[0,1], L_1)$, no.

Need to find $f \in A$ such that for all $\varepsilon > 0$ there exists $g \notin A$ and $d(f,g) < \varepsilon$.

$f = 1 \in A$





$\|f - g_n\|_1 \leq \frac{2}{n}$ each $g_n \notin A$.

Lemma: Suppose $A \subseteq X$ is not open. Then there is $x \in A$ and a sequence in A^c converging to x.

Pf: Since A is not open, there exists $x \in A$ such that for all $\varepsilon > 0$ $\underbrace{B_\varepsilon(x) \cap A^c \neq \phi}_{B_\varepsilon(x) \not\subseteq A}$. Thus, for each $n \in \mathbb{N}$ we can pick $x_n \in A^c$ with $d(x, x_n) < \frac{1}{n}$.

Then $d(x, x_n) \to 0$ and therefore $x_n \to x$.

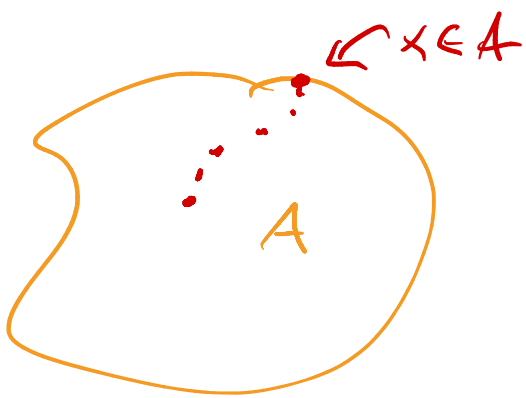
Def: A set $A \subseteq X$ is <u>closed</u> if A^c is open.

A is open: $\forall x \in A \ \exists r > 0$ s.t. $B_r(x) \subseteq A$

$\exists x \in A$ such that $\forall \varepsilon > 0$, $B_\varepsilon(x) \not\subseteq A$

Prop: (Sequential characterization of closed sets)

A set A is closed if and only if whenever (x_n) is a sequence in A converges to some x, $x \in A$.



Pf: Suppose A is closed and $y \notin A$. We will show there is no sequence in A converging to y. Since A^c is open there exists $B_\varepsilon(y) \subseteq A^c$. Any sequence in X converging to y contains terms in $B_\varepsilon(y)$ and is therefore not contained in A.

Suppose A is not closed. Since A is not closed,

A^c is not open and by the lemma above, there exists a sequence in $\underbrace{(A^c)^c}_{A}$ converging to some $x \notin A$.

Exercises: An arbitrary union of open sets is open.

An arbitrary intersection of closed sets is closed.

demorgans Laws

An arbitrary intersection of open sets need not be open

$A_n = (-1/n, \frac{1}{n})$ $\qquad \bigcap A_n = \{0\} \leftarrow$ not open

Exercise: A finite intersection of open sets is open

A finite union of closed sets is closed.