What about ℓ_p? Given $x, y \in \ell_p$ $1 \le p \le \infty$

$$\|x+y\|_p \le \|x\|_p + \|y\|_p$$

First observation: given $x, y \in \ell_p$ $x+y \in \ell_p$

$$|x_k + y_k| \le 2 \max(|x_k|, |y_k|)$$

$$|x_k + y_k|^p \le 2^p \max(|x_k|^p, |y_k|^p)$$

$\max(3,5) \le 3+5$

$$\le 2^p (|x_k|^p + |y_k|^p)$$

$$\underbrace{\sum_{k=1}^{\infty} |x_k + y_k|^p}_{\|x+y\|_p^p} \le 2^p (\|x\|_p^p + \|y\|^p)$$

Recall CS ineq:

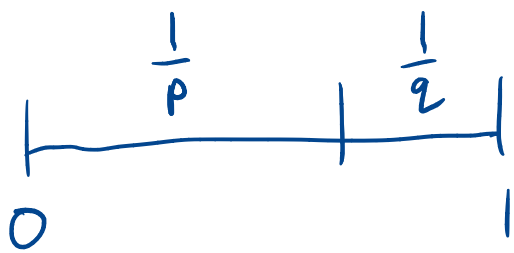
$x, y \in \ell_2$

$$q = \frac{p}{p-1}$$

$$\sum_{k=1}^{\infty} |x_k y_k| \leq \|x\|_2 \|y\|_2$$

Hölder's Inequality

Thm: Suppose $1 < p < \infty$ and q satisfies $\frac{1}{p} + \frac{1}{q} = 1$.



If $x \in \ell_p$ and $y \in \ell_q$ then

$$\sum_{k=1}^{\infty} |x_k y_k| \leq \|x\|_p \|y\|_q .$$

Triangle Inequality from Hölder's Ineq. $(1 < p < \infty)$

$$\|x+y\|_p^p = \sum_{k=1}^{\infty} |x_k + y_k|^p = \sum_{k=1}^{\infty} |x_k+y_k|\,|x_k+y_k|^{p-1}$$

$$\leq \sum_{k=1}^{\infty} |x_k|\,|x_k+y_k|^{p-1} + \sum_{k=1}^{\infty} |y_k|\,|x_k+y_k|^{p-1}$$

Hölder's Ineq!

$$\leq \|x\|_p \left[\sum_{k=1}^{\infty} |x_k+y_k|^{(p-1)q}\right]^{1/q} + \|y\|_p \left[\sum_{k=1}^{\infty} |x_k+y_k|^{(p-1)q}\right]^{1/q}$$

$$\boxed{\begin{array}{l}(p-1)q = p \\ \frac{1}{p}+\frac{1}{q} = 1\end{array}}$$

$$\leq \|x\|_p \|x+y\|_p^{p/q} + \|y\|_p \|x+y\|_p^{p/q}$$

Hence: $\|x+y\|_p^{p-p/q} \leq \|x\|_p + \|y\|_p$.

But $p - \frac{p}{q} = 1$ and the triangle inequality holds.

Hölder's Ineq is a consequence of Young's Inequality:

Given $a, b \geq 0$, $p > 1$ and q with $\frac{1}{p} + \frac{1}{q} = 1$

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

$a^p = \alpha \qquad b^q = \beta$

with equality iff $a = b^{p-1}$.

$$\alpha^{1/p} \beta^{1/q} \leq \frac{\alpha}{p} + \frac{\beta}{q}$$

Exercise: Prove Hölder from Young.

Def: Let (x_n) be a sequence in a metric space X.

We say $x_n \to x$ ((x_n) converges to x) if

for all $\varepsilon > 0$ there exists N such that

if $n \geq N$ then $d(x_n, x) < \varepsilon$.

$|x - x_n| <$

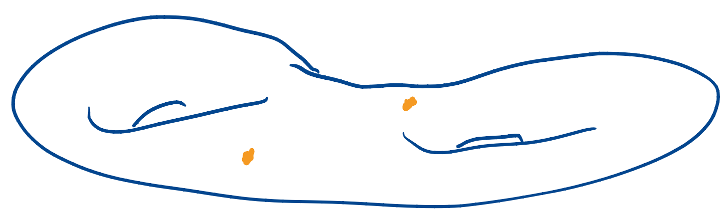
$d_p(x,y) = \|x-y\|_p$

Def: A sequence (x_n) in X is <u>Cauchy</u> if

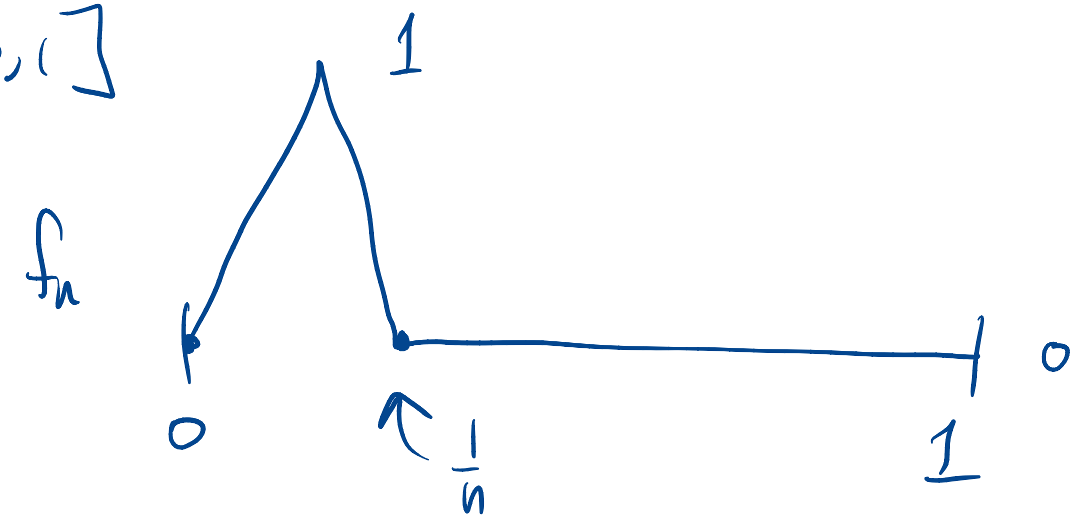
for all $\varepsilon > 0$ there exists N such that if $n, m \geq N$

$d(x_n, x_m) < \varepsilon$.

$|x_n - x_m|$



$C[0,1]$



Does $f_n \to 0$.

Answer depends on the norm on $C[0,1]$.

$\|f_n\|_\infty = 1$ for all n.

$\|f_n - 0\|_\infty = 1 \ \forall n$.

$d_\infty(f_n, 0) = 1$

But $f_n \to 0$ w.r.t. L_1 norm,

$$\| f \|_1 = \int_0^1 |f(x)| \, dx$$ ← def of L_1 norm from last week

$$\| f_n \|_1 = \frac{1}{2} \cdot \frac{1}{n} \cdot 1 = \frac{1}{2n} \to 0.$$

$$d_1(f_n, 0) \to 0 \quad \left(\text{Exercise: } f_n \to 0 \right)$$

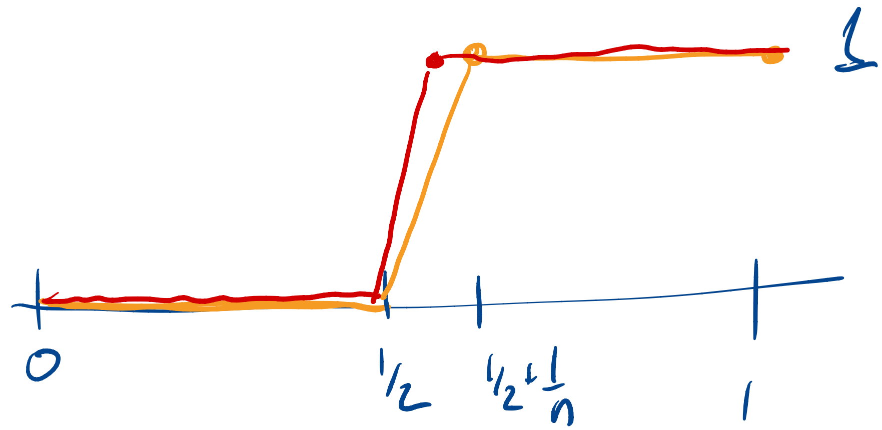
Exercise: Determine if $f_n \to 0$ in L_p $1 < p < \infty$.

Exercise: Suppose (x_n) is a sequence in a metric space converging to x. Then (x_n) is Cauchy.

Exercise: Show (f_n) does not have a limit in L_∞. (Show the sequence is not Cauchy).

Consider $\qquad\qquad\qquad\qquad (C[0,1], L_1)$

g_n



ε

Are the g_n's Cauchy in L_1?

If $n, m \geq N$ $\quad g_n(x) - g_m(x) = 0$ if $x \leq \frac{1}{2}$ or $x \geq \frac{1}{2} + \frac{1}{N}$

$$\| g_n - g_m \|_1 = \int_{1/2}^{1/2 + 1/N} |g_n(x) - g_m(x)| \, dx \leq \int_{1/2}^{1/2+\frac{1}{N}} 2 \, dx = \frac{2}{N}$$

Exercise: Show that there does not exist some $g \in C[0,1]$ such that $g_n \longrightarrow g$ (w.r.t. the L_1 norm).