Norms on $\mathbb{R}^2$

ℓ^1 $\quad \ell_1$: $\|x\|_1 = |x_1| + |x_2|$

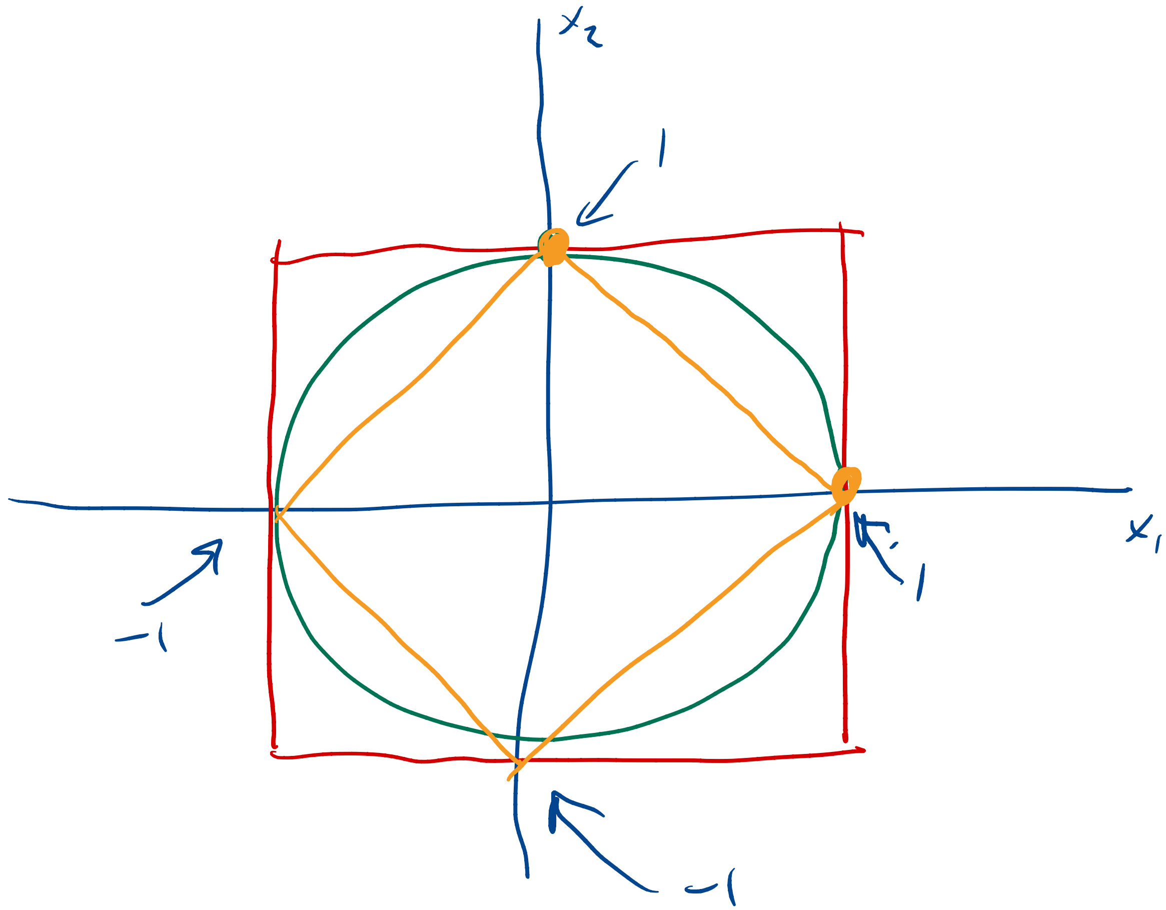
ℓ_2 : $\|x\|_2 = \left((x_1)^2 + (x_2)^2 \right)^{1/2}$

ℓ_∞ : $\|x\|_\infty = \max\left(|x_1|, |x_2| \right)$

Def:

Given a norm, the closed unit ball is

$$\overline{B}_1 = \{ x \in V : \|x\| \leq 1 \}$$



$\|x\|_\infty \le 1$

$\max(|x_1|, |x_2|) \le 1$

$=$

$\|x\|_1 \le 1$

$\underline{|x_1| + |x_2| = 1}$

$x_1 + x_2 = 1$

$x \quad y \quad y = 1 - x$

$\|x\|_2 \le 1$

$(x_1^2 + x_2^2)^{1/2} \le 1$

$x_1^2 + x_2^2 \le 1$

More generally $\quad 1 \le p < \infty \quad p \in \mathbb{R}$

$$\|x\|_p = \underline{\left(|x_1|^p + |x_2|^p\right)^{1/p}}$$

$$\|cx\|_p = |c|\,\|x\|_p$$

More more generally: $\quad (\mathbb{R}^n, \ell_p)$

ℓ_p applies to $\mathbb{R}^n$

Def: Given $1 \le p < \infty$, ℓ_p is the set of sequences $x = (x_n)$ with $\underbrace{\left[\sum_{n=1}^{\infty} |x_n|^p\right]^{1/p}} < \infty$,

in which case $\|x\|_p = \underline{\hspace{4cm}}$

$x = (x_n)$

$$\|x\|_\infty = \sup_{n=1}^{\infty} |x_n|$$

$\hookrightarrow$ l_∞ : set of bounded sequences

A sequence is a function $\mathbb{N} \to \mathbb{R}$.

$$k \longmapsto x_k$$

continuous functions on $[0,1]$

Norms on $C[0,1]$ $\quad f : [0,1] \to \mathbb{R}$

$$\|f\|_1 = \int_0^1 |f(x)|\, dx \qquad \sum_{k=1}^{\infty} |x_k|$$

$$\|f\|_2 = \left[\int_0^1 |f(x)|^2\, dx\right]^{1/2}$$

$$\|f\|_p = \left[\int_0^1 |f(x)|^p \, dx\right]^{1/p} \qquad \left[\sum_{k=1}^{\infty} |x_k|^p\right]^{1/p}$$

$$\|f\|_\infty = \sup_{x \in [0,1]} |f(x)| \leftarrow \text{default}$$

Triangle inequalities!

$(\mathbb{R}^2, \ell_2)$

1. Prop (arithmetic-geometric mean inequality special case)

For all $a, b \in \mathbb{R}$

$$|a|\,|b| \leq \frac{1}{2}\left(a^2 + b^2\right).$$

Pf: It suffices to assume $a \geq 0$ and $b \geq 0$.

Observe $(a-b)^2 \geq 0$ and

$$(a-b)^2 = a^2 - 2ab + b^2.$$

Hence $a^2 - 2ab + b^2 \geq 0$

and the inequality follows from a computation. $\square$

$\alpha = a^2$ $\quad$ $\beta = b^2$

$\alpha^{1/2} = a$ $\quad$ $\beta^{1/2} = b$

$$ab \leq \frac{1}{2}(a^2 + b^2)$$

$$\alpha^{1/2}\beta^{1/2} \leq \frac{1}{2}(\alpha + \beta)$$

$$(\alpha\beta)^{1/2}$$

(2) Prop Cauchy-Schwartz Inequality

For all $x, y \in \mathbb{R}^n$ $\qquad x \cdot y = \|x\| \|y\| \cos\theta$

$$\underbrace{\sum_{k=1}^{n} |x_k y_k|}_{\downarrow} \le \|x\|_2 \|y\|_2$$

$$\left| \sum_{k=1}^{n} x_k y_k \right|$$

$x_1 y_1 + x_2 y_2 + x_3 y_3$

(x_1, x_2, x_3)

(y_1, y_2, y_3)

Pf: Suppose first that $\|x\|_2 = 1$, $\|y\|_2 = 1$.

Then $\displaystyle\sum_{k=1}^{n} |x_k y_k| \le \frac{1}{2} \sum_{k=1}^{n} \left(|x_k|^2 + |y_k|^2 \right) = 1 = \|x\|_2 \|y\|_2$.

The inequality is trivial if $x = 0$ or $y = 0$.

Now suppose $x \neq 0, y \neq 0$ are arbitrary.

Let $z = \frac{x}{\|x\|_2}$ $w = \frac{y}{\|y\|_2}$. Observe $\|z\|_2 = 1$ and $\|w\|_2 = 1$.

So $\sum_{k=1}^{n} |z_k w_k| \leq \|z\|_2 \|w\|_2$.

Equivalently $\sum_{k=1}^{n} \frac{|x_k y_k|}{\|x\|_2 \|y\|_2} \leq 1$ and the result follows.

$\square$

Prop (CS inequality for ℓ_2)

For all $x, y \in \ell_2$ $\sum_{k=1}^{\infty} |x_k y_k| \leq \|x\|_2 \|y\|_2$.

Pf: For any N, by the CS-inequality on $\mathbb{R}^N$

$$\sum_{k=1}^{N} |x_k y_k| \leq \left[\sum_{k=1}^{N} |x_k|^2\right]^{1/2} \left[\sum_{k=1}^{N} |y_k|^2\right]^{1/2}$$

$$\leq \|x\|_2 \|y\|_2 .$$

Now take a limit in N. $\square$

$c_k \geq 0$

$$\sum_{k=1}^{N} c_k \leq M$$

$x = (x_1, x_2, x_3, \ldots x_N | \ldots$

$(x_1, x_2, x_3, \ldots, x_N) \in \mathbb{R}^N$

Cor: For all $x, y \in \ell_2$ $x+y \in \ell_2$ and

$$\|x+y\|_2 \leq \|x\|_2 + \|y\|_2 .$$

$$\|x\|_2 = \left[\sum_{k=1}^{\infty} x_k^2\right]^{1/2}$$

Pf: For each k,

$$(x_k + y_k)^2 = (x_k)^2 + 2x_k y_k + (y_k)^2.$$

Hence
$$\sum_{k=1}^{\infty} (x_k + y_k)^2 = \|x\|_2^2 + 2\sum_{k=1}^{\infty} x_k y_k + \|y\|_2^2 .$$

$$\leq \|x\|_2^2 + 2\|x\|_2 \|y\|_2 + \|y\|_2^2$$

$$= \left(\|x\|_2 + \|y\|_2\right)^2 .$$

Now apply square roots. $\square$