$f_1 : \mathbb{N} \to A$

$f_2 : \mathbb{N} \to B$, sorjectios $(A, B \neq \emptyset)$

$f : \mathbb{N} \times \{0,1\} \to A \cup B$

$f(n, k) = f_k(n)$ is a sorjection

$\mathbb{N} \times \{0,1\} \subseteq \mathbb{N} \times \mathbb{N}$ and is hnce countable.

$f : \mathbb{N} \times \{0,1\} \to A \cup B$ is

a sorjection from a countable set.

$\bigcup_{k=1}^{\infty} A_k$ $\quad A_k$ countable.

$A_1, A_2, \ldots, A_n$ all countable

$\bigcup_{k=1}^{n} A_k$ is countable. $\boxed{\forall n}$

$f_k : \mathbb{N} \to A_k$, surjection

$f : \mathbb{N} \times \mathbb{N} \to \bigcup A_k$

$f(n,k) = f_k(n)$

Cor: $\mathbb{Q}$ is countable $\mathbb{Q} = \mathbb{Q}_- \cup \{0\} \cup \mathbb{Q}_+$

Theorem: $\mathbb{R}$ is uncountable

Pf: It is enough to show that $[0,1]$ is uncountable.

Let (x_n) be a sequence in $[0,1]$. We will show the sequence does not exhaust all of $[0,1]$.

Let us write

$$
\begin{aligned}
x_1 &= 0.\boxed{a_{11}}\, a_{12}\, a_{13} \cdots \qquad \text{(base 10)} \\
x_2 &= 0.\, a_{21}\, \boxed{a_{22}} \cdots \\
x_3 &= 0.\, a_{31}\, a_{32}\, \boxed{a_{33}} \cdots \\
&\vdots
\end{aligned}
$$

In the case some x_n admits two expansions, pick the terminating one. Let

$$
b_k = \begin{cases} 7 & \text{if } a_{kk} \neq 7 \\ 3 & \text{if } a_{kk} = 7. \end{cases}
$$

Consider $x = 0.b_1 b_2 b_3 \dots \in [0,1]$.

It has a unique expansion because no digit is 0 or 9.

Moreover $x \neq x_k$ for all k since x has a unique expansion and since $b_k \neq a_{kk}$.

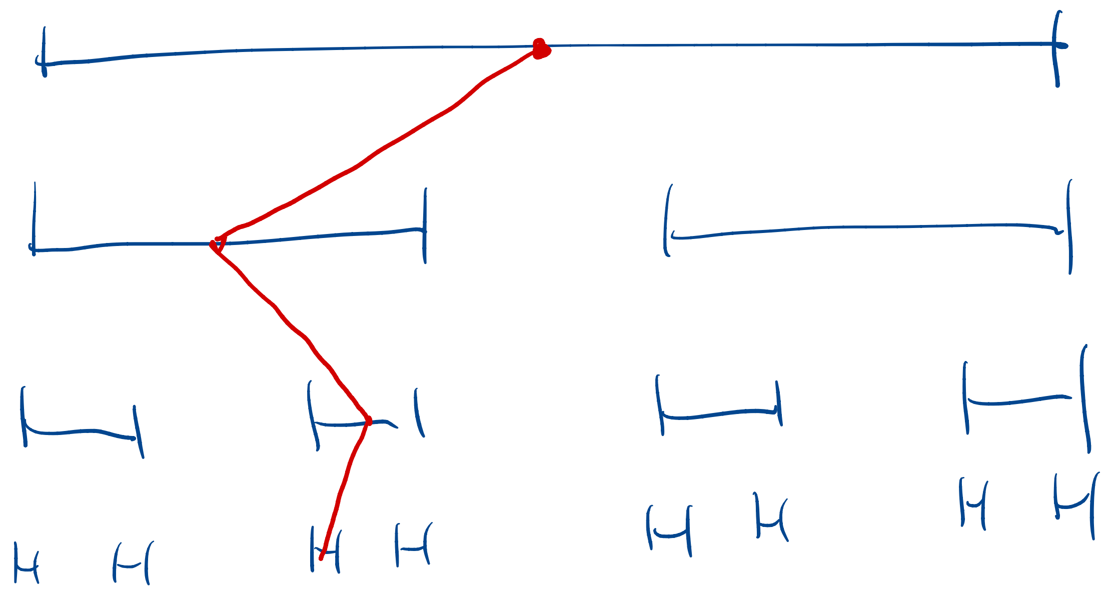
$\mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ $\qquad$ $\mathbb{N}^k$ is countable $\forall k \in \mathbb{N}$

$$\mathbb{N}^\omega$$

$\mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \dots$

$\overset{L\ R}{\{0,1\}} \times \{0,1\} \times \{0,1\} \times \dots$ $\qquad$ Is this countable?

Exercise: Use Cantor's diagonal trick to show this is not countable.



Each $x \in \Delta$ admits a unique base 3 expansion with no 1s.

$\ldots 122 \ldots$

$\ldots 200 \ldots$

Map: $\Delta \xrightarrow{F} [0,1]$, Cantor function

$$x = \sum_{k=1}^{\infty} \frac{2a_k}{3^k}$$ where each $a_k \in \{0,1\}$

$$F(x) = \sum_{k=1}^{\infty} \frac{a_k}{2^k}$$

This map is clearly onto. Since $[0,1]$ is uncountable, so is Δ.

$F(1/3)$ $\quad \frac{1}{3} = 0.022\cdots\cdots$ (base 3)

$$F(1/3) = 0.011\ldots \quad (\text{base } 2)$$

$$= \underbrace{0.10\ldots\ldots}_{1/2} \quad (\text{base } 2)$$

$F(2/3)$

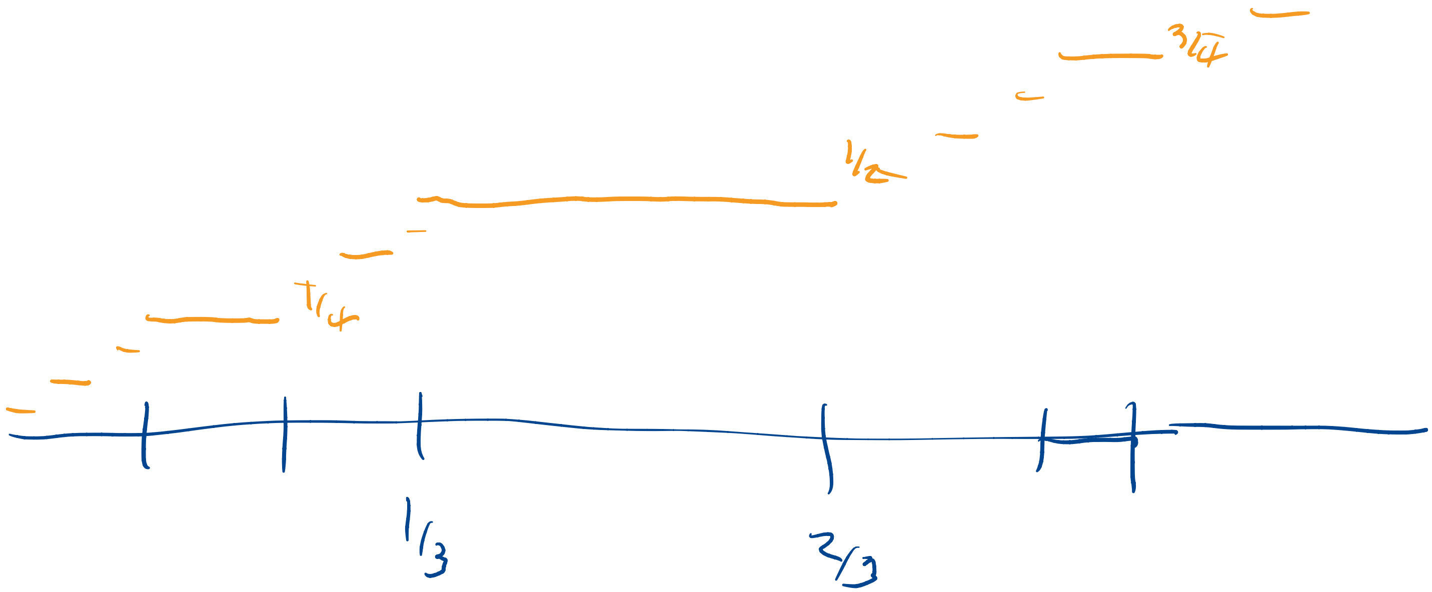
$$\frac{2}{3} = 0.200\ldots \quad (\text{base } 3)$$

$$F(2/3) = \underbrace{0.100\ldots\ldots}_{1/2} \quad (\text{base } 2)$$

Exercise: F is increasing.

We can extend F to all of $[0,1]$ by

$$F(x) = \sup\{F(y) : y \in \Delta,\ y \leq x\}$$



Metric spaces:

X set. A metric on X is a function

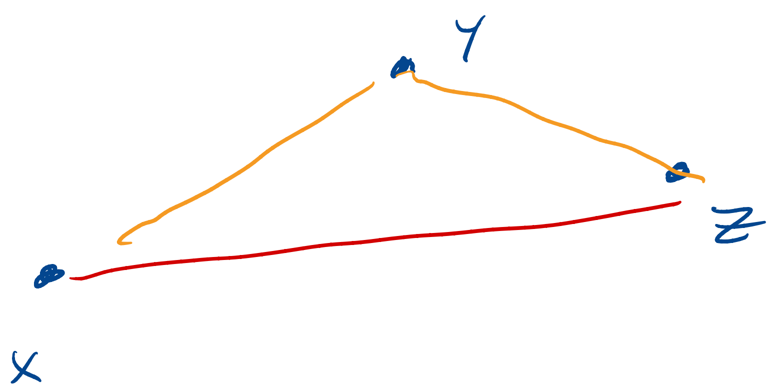
$d : X \times X \to \mathbb{R}$ such that

1) $d(x,y) \geq 0$

2) $d(x,y) = 0 \iff x = y$

3) $d(x,y) = d(y,x)$

4) $\underbrace{d(x,z) \leq d(x,y) + d(y,z) \quad \forall x,y,z \in X}_{\text{Triangle inequality}}$

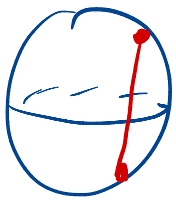


A set equipped with a metric is called a metric space.

e.g. $\mathbb{R}$ $\quad d(x,y) = |x-y|$

$\mathbb{R}^3$ $\quad d(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2}$

$$S^2 = \{x \in \mathbb{R}^3 : d(x,0) = 1\}$$



$d(x,y) =$

Any subset of a metric space is a metric space.

$C[0,1] \to$ continuous functions on $[0,1]$

$$d(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|$$

$$= \max_{x \in [0,1]} |f(x) - g(x)|$$

Δ ineq:

$$|f(x) - g(x)| \leq |f(x) - h(x)| + |h(x) - g(x)|$$

$$\leq d(f,h) + d(h,g)$$

$$\underbrace{\sup_{x \in [0,1]} |f(x) - g(x)|}_{d(f,g)} \leq d(f,h) + d(h,g)$$

Related notion: norm on a real vector space.

A norm on a vector space V is a function

$$\|\cdot\| : V \to \mathbb{R} \quad \text{such that}$$

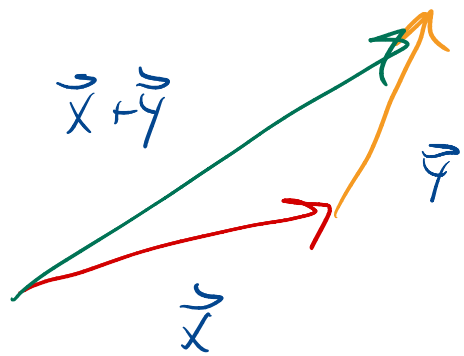
1) $\|x\| \geq 0 \quad \forall x \in V$

2) $\|x\| = 0 \iff x = 0$

3) $\|\alpha x\| = |\alpha| \, \|x\| \quad \forall \alpha \in \mathbb{R}, x \in V$

4) $\underbrace{\|x+y\| \leq \|x\| + \|y\|}_{\text{triangle inequality}}$

(a norm tells you distance from 0)



Given a norm we have an associated metric

$$d(x,y) = ||x-y|| \quad \Big| \quad d(x,0) = ||x-0|| = ||x||$$

Each of the earlier metrics ~~norms~~ on $\mathbb{R}$, $\mathbb{R}^3$ and $C[0,1]$ arise from a norm.

If $x \in \mathbb{R}^3$, $\quad ||x|| = \sqrt{x_1^2 + x_2^2 + x_3^2}$

$$C[0,1] \qquad ||f|| = \sup_{x \in [0,1]} |f(x)| = \max_{x \in [0,1]} |f(x)|$$