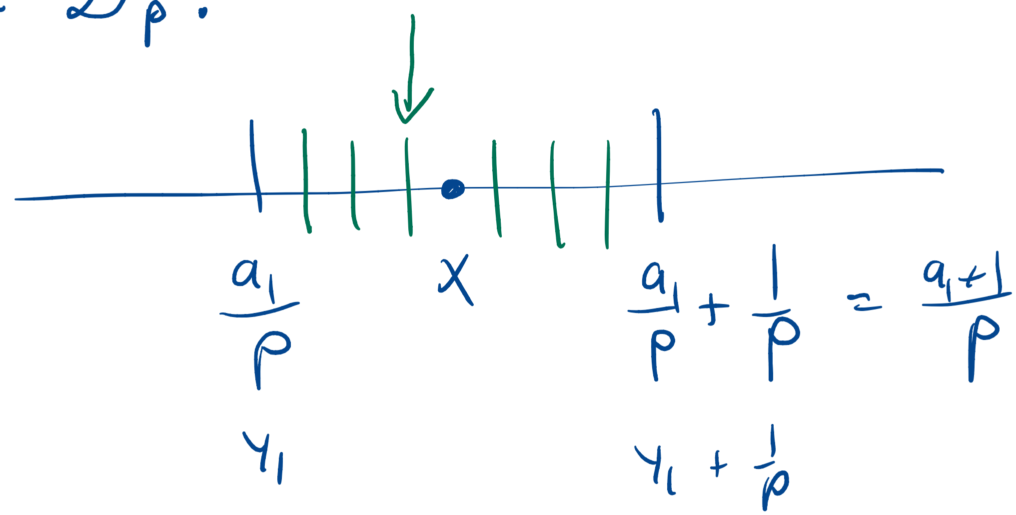
Prop: Each $x \in [0,1]$ admits a base p expansion.

Pf: Case $x = 0$ is trivial.

Suppose $0 < x \leq 1$.

Let $a_1 = \max \{ d \in \mathbb{N}_{\geq 0} : \frac{d}{p} < x \}$ and

observe $a_1 \in \mathcal{D}_p$.



Let $y_1 = \frac{a_1}{p}$ and observe $y_1 < x \leq y_1 + \frac{1}{p}$.

Now: let $a_2 = \max \{ d \in \mathbb{N}_{\geq 0} : y_1 + \frac{d}{p^2} < x \}$

and observe $a_2 \in \mathcal{D}_p$. $\left(y_1 + \frac{p}{p^2} = y_1 + \frac{1}{p} \geq x \right)$

Let $y_2 = y_1 + \frac{a_2}{p^2}$.

Inductively we can select digits $a_1, a_2, \ldots$ each in $\mathcal{D}_p$

and approximations satisfying

$$y_k = \sum_{j=1}^{k} \frac{a_k}{p^k}$$

$$-\frac{1}{p^k} \leq x - y_k \leq \frac{1}{p^k}$$

$$y_k < x \leq y_k + \frac{1}{p^k}.$$

Observe $|x - y_k| \leq \frac{1}{p^k}$. Since $\left(\frac{1}{p}\right)^k \to 0$

$y_k \to x$ by the squeeze theorem, $\square$

$$\boxed{\sum_{k=1}^{\infty} \frac{a_k}{p^k} = x}$$

$$\sum_{k=1}^{\infty} b_k = z$$

$$\lim_{n\to\infty} \boxed{\sum_{k=1}^{n} b_k} = z$$

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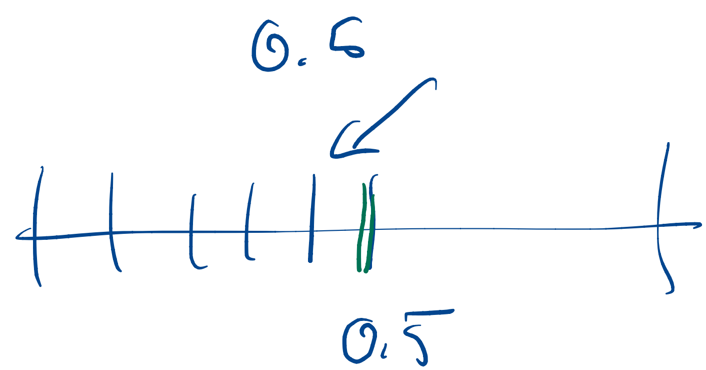
base 10 $\quad x = \frac{1}{2}$

$0.4999\ldots$



ON HW: Given $x \in (0,1)$ exactly one of the following is true

1) x admits a unique base p expansion, and the expansion does not end in a trail of 0's or $(p-1)$'s.

2) x admits exactly two expansions.

One is $0.a_1 \cdots a_N 0 0 \cdots$ (base p)

with $a_N \neq 0$. The other is

$$0.\underbrace{a_1 \cdots a_{N-1}}(a_N - 1)(p-1)(p-1)\cdots \quad (\text{base } p).$$

$0.5000\cdots$

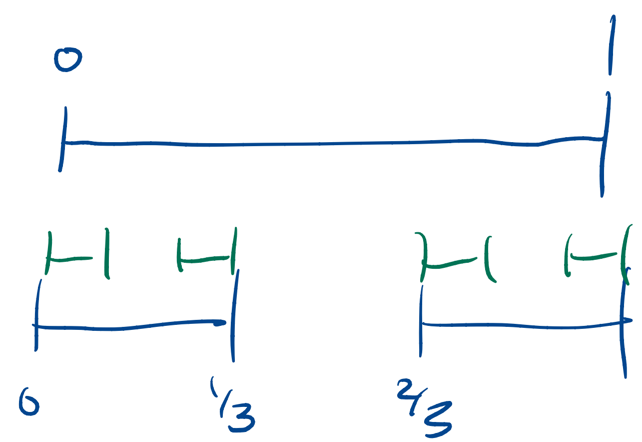
$0.4999\cdots$

Cantor Set

$A_0 = [0,1]$

$A_1 = [0,\frac{1}{3}] \cup [\frac{2}{2}, 1]$

$= \frac{1}{3}A_0 \cup \left(\frac{2}{3} + \frac{1}{3}A_0\right)$

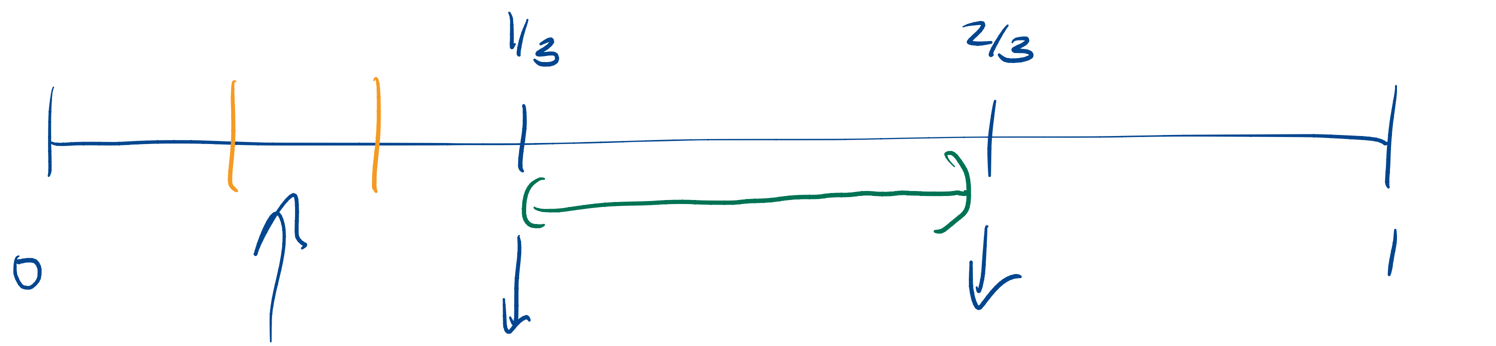


$$A_2 = [0,\tfrac{1}{9}] \cup [\tfrac{2}{9},\tfrac{1}{3}] \cup [\tfrac{2}{3},\tfrac{7}{9}] \cup [\tfrac{8}{9},1]$$

$$= \tfrac{1}{3}A_1 \cup \left(\tfrac{2}{3} + \tfrac{1}{3}A_1\right)$$

In general $A_{k+1} = \frac{1}{3}A_k \cup \left(\frac{2}{3} + \frac{1}{3}A_k\right)$.

Cantor set $\Delta = \bigcap A_k$.



$0.10\cdots$ (base 3) $\qquad$ $0.12\cdots$ (base 3)

$0.02\cdots$ (base 3) $\qquad$ $0.20\cdots0$ (base 3)

A_1 is the subset of $[0,1]$ where the number admits a base 3 expansion starting with 0 or 2

A_2 is the subset of $[0,1]$ where the number admits a base 3 - - - - - with first two digits equal to 0 or 2.

$$\Delta = \{ x \in [0,1] : x \text{ admits a base 3 expansion with no 1's} \}$$

Is the Cantor set big or small?

Answer #1 From $[0, 1]$ we remove

1 interval of length $1/3$

2 intervals of length $1/3^2$

4 intervals of length $1/3^3$

$\vdots$

2^{k-1} intervals of length $1/3^k$

$$\frac{1}{3} + 2\cdot\left(\frac{1}{3^2}\right) + 2^2\left(\frac{1}{3^3}\right) + \cdots + 2^{k-1}\left(\frac{1}{3^k}\right) + \cdots$$

$$\frac{1}{2}\left(\frac{2}{3}\right) + \frac{1}{2}\left(\frac{2}{3}\right)^2 + \frac{1}{2}\left(\frac{2}{3}\right)^3 + \cdots + \frac{1}{2}\left(\frac{2}{3}\right)^k + \cdots$$

$$\sum_{k=1}^{\infty} \frac{1}{2}\left(\frac{2}{3}\right)^k = \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k = \frac{1}{2}\,\frac{2/3}{1-2/3} = \frac{1}{2}\,\frac{2}{3-2} = 1$$