How is $\mathbb{R}$ different from $\mathbb{Q}$?

What is an upper bound for $\emptyset$?

Completeness

Axiom of Completeness:

Every nonempty subset of $\mathbb{R}$ that is bounded above has a supremum (least upper bound).

Recall: $b \in \mathbb{R}$ is a supremum of $A \subseteq \mathbb{R}$

if a) b is an upper bound for A, i.e.

for all $a \in A$, $a \leq b$.

b) whenever b' is an upper bound for A,

$$b \leq b' \qquad \text{(leastness)}$$

Consequences:

1) Cauchy criterion

2) Bolzano-Weierstrass Thm

3) Monotone Conv. Thm.

4) Nested interval Property

Extended Real Numbers

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$$

Rules

$$\infty \geq x \qquad \forall x \in \overline{\mathbb{R}}$$

$$-\infty \leq x$$

$x + \infty = \infty$ so long as $x \neq -\infty$

$x + -\infty = -\infty$ " " " " $x \neq \infty$

$\infty + (-\infty)$ is not allowed.

$$a \cdot \infty = \begin{cases} \infty & a > 0 \\ -\infty & a < 0 \\ \text{illegal} & a = 0 \end{cases} \quad \text{(for now)}$$

If we work with $\overline{\mathbb{R}}$ we can take the infimum / supremum of any set.

If $A \subseteq \mathbb{R}$ and is not bounded above in $\mathbb{R}$

$$\sup A = \infty$$

$$\sup \emptyset = -\infty$$

Recall $\lim_{n \to \infty} x_n = L$ if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n \geq N$ then $|x_n - L| < \varepsilon$.

Limits of sequences need not exist.

$$x_n = n$$

$$x_n = (-1)^n$$

$x_n = r_n$, r_n is an enumeration of $\mathbb{Q} \cap [0,1]$

One can make mathematical errors if one assumes limits exist before establishing this is the case.

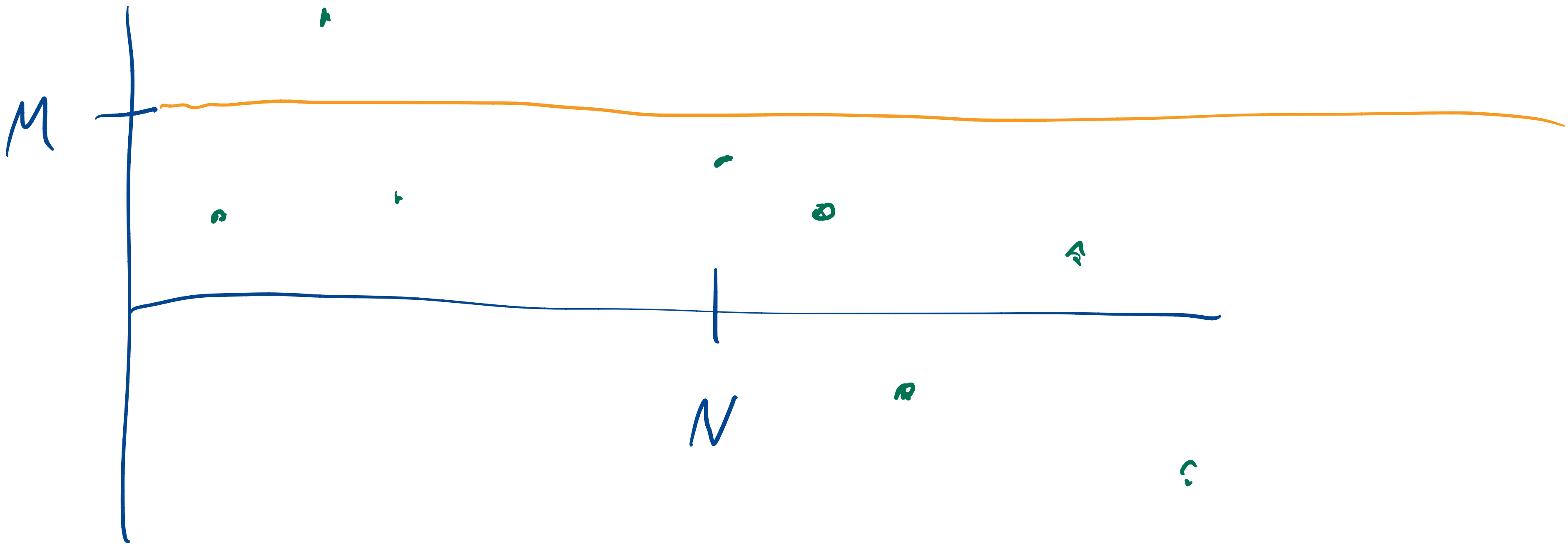
$$\lim_{n \to \infty} x_n$$

There are related concepts, limit infimum / supremum that always exist (in $\overline{\mathbb{R}}$)

Version A:

We say $M \in \overline{\mathbb{R}}$ is an <u>eventual upper bound</u> for a sequence $\{x_n\}$ in $\overline{\mathbb{R}}$ if there exists N such that

$$x_n \leq M \quad \text{for all} \quad n \geq N.$$



Def: $\limsup_{n\to\infty} x_n = \inf\{M : M \text{ is an e.u.b.}\}$

$x_n \in \overline{\mathbb{R}}$

E.g. $x_n = (-1)^n$

$\limsup_{n \to \infty} x_n = 1$

$M = 1$ is an e.u.b.

Any $M < 1$ is not an e.u.b:

given any N either $x_N = 1$ or $x_{N+1} = 1$

and there is $n \geq N$ with $x_n > M$.

Any $M > 1$ is an e.u.b.

Set of e.u.b: $[1, \infty]$

$\inf [1, \infty] = 1.$