MATH 253
Quiz 8

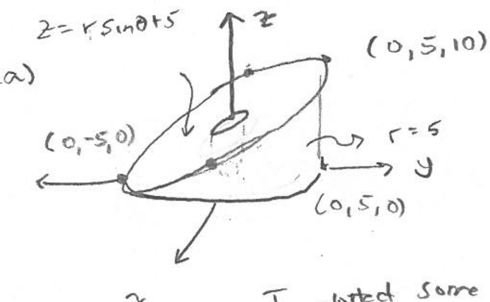
Name: SOLUTIONS

Due at 11:59 pm on April 2, 2021

Instructions: (10 points total) Show all work for credit. You may use your book, but no other resource. **GS:** Scan TWO pages for your solutions.

1. (4 pts.) Consider the solid E which, in cylindrical coordinates, is bounded by the planes $z = 0$, $z = r\sin(\theta) + 5$ and the cylinders $r = 1$ and $r = 5$
   (a) Sketch (as best you can) the solid E.
   (b) Compute the definite integral $\iiint_E x - y\,dV$

(a)



Top: $z = y+5$

I plotted some points for reference

$(5,0,\pm 5)$ are dots on top.

E: $0 \le z \le r\sin\theta + 5$

$0 \le \theta \le 2\pi$

$1 \le r \le 5$

$$\iiint_E x-y\,dV = \int_0^{2\pi}\int_1^5\int_0^{r\sin\theta+5} (r\cos\theta - r\sin\theta)\,r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^5 (r^2\cos\theta - r^2\sin\theta)(r\sin\theta+5)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^5 r^2(r\sin\theta+5)(\cos\theta-\sin\theta)\,dr\,d\theta = \int_0^{2\pi}(\cos\theta-\sin\theta)\int_1^5 r^3\sin\theta + 5r^2\,dr\;d\theta$$

$$= \int_0^{2\pi}(\cos\theta-\sin\theta)\left[\frac{1}{4}r^4\sin\theta + \frac{5}{3}r^3\right]_1^5 d\theta = \int_0^{2\pi}(\cos\theta-\sin\theta)\left[\left(\frac{625}{4}\sin\theta + \frac{625}{3}\right) - \left(\frac{1}{4}\sin\theta + \frac{5}{3}\right)\right]d\theta$$

$$= \int_0^{2\pi}(\cos\theta-\sin\theta)\left(156\sin t + \frac{620}{3}\right)d\theta = \int_0^{2\pi} 156\cos t\sin t + \frac{620}{3}\cos\theta - 156\sin^2 t - \frac{620}{3}\sin t\,dt$$

↓ double angle

$$= \frac{156\sin^2 t}{2} + \frac{620}{3}\sin t - 156\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right] + \frac{620}{3}\cos t\Big|_0^{2\pi}$$

$$= 78\sin^2 t\Big|_0^{2\pi} - \frac{620}{3}\sin t\Big|_0^{2\pi} - \frac{156\theta}{2}\Big|_0^{2\pi} + \frac{156\sin 2\theta}{4}\Big|_0^{2\pi} + \frac{620}{3}\cos t\Big|_0^{2\pi}$$

$$0 \quad - \quad 0 \quad -156\pi \quad + \quad 0 \quad + \quad 0$$

$$= \boxed{-156\pi}$$

2. (2 pts.) Use the 'Change of Variable' ideas to show that the volume element dV in cylindrical coordinates is

$$dV = r\,dr\,d\theta\,dz.$$

Show your work, including the formulas for the transformation and the determinant of the Jacobian.

Transformation: $x = r\cos\theta \quad y = r\sin\theta \quad z = z.$

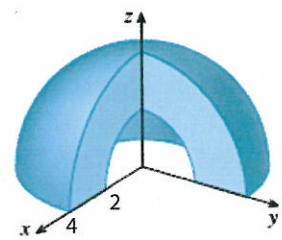
$$\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

so

$dV = r\,dr\,d\theta\,dz$

3. (4 pts.) Pictured is a solid B that fills up three-quarters of the region between hemispheres of radius 2 and one of radius 4.

(a) Without doing any calculus at all, compute the volume of the solid B. (You may look up the volume of a sphere if you do not remember it.)



$\text{Vol} = \frac{4}{3}\pi R^3$ for sphere

$$\text{Vol} = \underbrace{\tfrac{1}{2}}_{\text{Hemisphere}}\underbrace{\left(\tfrac{3}{4}\right)}_{75\%}\left[\tfrac{4}{3}\pi(4)^3 - \tfrac{4}{3}\pi(2)^3\right] = \frac{\pi}{2}\left[64-8\right] = \boxed{28\pi}$$

(b) Now use spherical coordinates and an appropriate triple integral to compute this volume.

(Lots of variants!) $\quad \frac{\pi}{2} \le \theta \le 2\pi, \quad 2 \le r \le 4, \quad 0 \le \varphi \le \pi/2$

$$\text{Vol} = \iiint_B dV = \int_{\pi/2}^{2\pi}\int_2^4\int_0^{\pi/2} \rho^2 \sin\varphi \, d\varphi\, d\rho\, d\theta$$

$$= \int_{\pi/2}^{2\pi}\int_2^4 -\rho^2\cos\varphi \Big|_0^{\pi/2} d\rho\, d\theta = \int_{\pi/2}^{2\pi}\int_2^4 -\rho^2(0-1)\, d\rho\, d\theta = \int_{\pi/2}^{2\pi}\int_2^4 \rho^2\, d\rho\, d\theta$$

$$= \int_{\pi/2}^{2\pi} \frac{1}{3}\rho^3 \Big|_2^4 d\theta = \int_{\pi/2}^{2\pi} \frac{64}{3} - \frac{8}{3}\, d\theta = \int_{\pi/2}^{2\pi} \frac{56}{3}\, d\theta = \frac{56}{3}\left(2\pi - \frac{\pi}{2}\right)$$

$$= \frac{56}{3}\left(\frac{3\pi}{2}\right) = \boxed{28\pi}$$